Fractional quantum Hall effect in a trapping potential:

In this exercise, we will investigate the effect of a simple trapping potential $V_t(r) = \frac{1}{2}v_t r^2$ on the Laughlin states. We consider spinless electrons in two dimensions.

[This exercise is significantly dependent on Exercise 5 where we discussed the case without a trapping potential. The students need to frequently refer to the that while proceeding with this exercise.]

In exercise 5, we have introduced the operators

$$\hat{\Pi}_{x/y} = \hat{p}_{x/y} + \frac{e}{c}A_{x/y}. \tag{1}$$

1. Now, we define the following operators

$$\hat{\Pi}_x(\alpha) = \hat{p}_x - \frac{\alpha}{2}y \tag{2}$$

$$\hat{\Pi}_y(\alpha) = \hat{p}_y + \frac{\alpha}{2}x \tag{3}$$

$$\hat{X}(\alpha) = \hat{x} - \alpha^{-1}\hat{\Pi}_y(\alpha) \tag{4}$$

$$\hat{Y}(\alpha) = \hat{y} + \alpha^{-1}\hat{\Pi}_x(\alpha). \tag{5}$$

   Express $\hat{\Pi}_x^2$ as a function of $\hat{\Pi}_x^2(\alpha)$, $\hat{Y}(\alpha)^2$ and y^2.

   [Hint: You need to use the commutation relations derived in Exercise 5.]

   Solution: This computation can be cumbersome if not done appropriately. First we can note that

$$\hat{\Pi}_x^2 = (\hat{\Pi}_x(\alpha) + \frac{\alpha - m\omega_c}{2}y)^2 = \hat{\Pi}_x^2(\alpha) + \frac{(\alpha - m\omega_c)^2}{4}y^2 + (\alpha - m\omega_c)\hat{\Pi}_x y. \tag{6}$$

   We used the commutation of $\hat{\Pi}_x$ and y. Then, we can write

$$\hat{Y}(\alpha)^2 = y^2 + \alpha^{-2}\hat{\Pi}_x^2 + 2\alpha^{-1}y\hat{\Pi}_x(\alpha) \Leftrightarrow y\hat{\Pi}_x(\alpha) = \frac{\alpha}{2}(\hat{Y}(\alpha)^2 - y^2 - \alpha^{-2}\hat{\Pi}_x^2). \tag{7}$$

   We then obtain:

$$\hat{\Pi}_x^2 = \hat{\Pi}_x^2(\alpha) + \frac{(\alpha - m\omega_c)^2}{4}y^2 + (\alpha - m\omega_c)\frac{\alpha}{2}(\hat{Y}(\alpha)^2 - y^2 - \alpha^{-2}\hat{\Pi}_x^2) \tag{8}$$

$$= (1 - \frac{\alpha - m\omega_c}{2\alpha})\hat{\Pi}_x^2(\alpha) - \frac{(\alpha - m\omega_c)(\alpha + m\omega_c)}{4}y^2 + \frac{\alpha(\alpha - m\omega_c)}{2}\hat{Y}(\alpha)^2 \tag{9}$$

2. In the similar fashion, express $\hat{\Pi}_y^2$ as a function of $\hat{\Pi}_y^2(\alpha)$, $\hat{X}(\alpha)^2$ and x^2.

   Solution: We obtain the expression for $\hat{\Pi}_y$ by exchanging x and y, and X and Y.

3. Show that by choosing α appropriately, one can rewrite the full Hamiltonian, i.e.

$$H_{\text{Landau}} + V_t(r) = \frac{1}{2m}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2) + \frac{1}{2}v_t r^2,$$

in the form

$$\lambda_1 \hat{\Pi}_x(\alpha)^2 + \lambda_2 \hat{\Pi}_y(\alpha)^2 + \lambda_3 \hat{X}(\alpha)^2 + \lambda_4 \hat{Y}(\alpha)^2. \tag{10}$$

Find the expressions of the λ_j's.

Solution: We want the term in y^2 obtained in a) to cancel V_t. This gives us:

$$\alpha^2 = 4mv_t + m^2\omega_c^2 \Leftrightarrow \alpha = \sqrt{4mv_t + m^2\omega_c^2}. \tag{11}$$

Fixing α to this value, we obtain

$$\lambda_1 = \lambda_2 = \frac{1}{2m}\frac{\alpha + m\omega_c}{2\alpha} \tag{12}$$

$$\lambda_3 = \lambda_4 = \frac{1}{2m}\frac{\alpha(\alpha - m\omega_c)}{2} \tag{13}$$

4. First, convince yourself that the above Hamiltonian is made of two independent harmonic oscillators and thus, can be written as:

$$H_{\text{Landau}} + V_1 = \hbar\tilde{\omega}_c(a^\dagger a + \frac{1}{2}) + \hbar\omega_t(b^\dagger b + \frac{1}{2}) \tag{14}$$

Now, determine the frequencies $\tilde{\omega}_c$ and ω_t.

[Hint: You can use an analogy from Exercise 5.]

Solution: We verify that $[\hat{\Pi}_{x/y}, \hat{X}/\hat{Y}] = 0$. We also have:

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i\hbar\alpha \tag{15}$$

$$[\hat{X}, \hat{Y}] = i\hbar\alpha^{-1} \tag{16}$$

We therefore recognize two independent harmonic oscillators. To get the energy, we just need to remark that

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i\hbar m\omega_c \tag{17}$$

In the presence of the trapping field, we have:

$$\tilde{m}_c\tilde{\omega}_c = \alpha \text{ and } \frac{1}{2\tilde{m}_c} = \lambda_1 \tag{18}$$

We obtain

$$\tilde{m}_c = m(\frac{\alpha + m\omega_c}{2\alpha})^{-1} = \frac{2m\alpha}{\alpha + m\omega_c} \tag{19}$$

and therefore

$$\tilde{\omega}_c = \frac{\alpha + m\omega_c}{2}. \tag{20}$$

Similarly, for the second oscillator $\tilde{m}_t\tilde{\omega}_t = \alpha^{-1}$

$$\tilde{m}_t = m(\frac{\alpha(\alpha - m\omega_c)}{2})^{-1} = \frac{2m}{\alpha(\alpha - m\omega_c)}. \tag{21}$$

$$\tilde{\omega}_t = \frac{\alpha - m\omega_c}{2}. \tag{22}$$

5. What is ω_t at $v_t = 0$? What does it signify?

Solution: When the trapping potential disappear, the effective mass $\tilde{m}_t$ diverges, signifying the rigidity of the effective Harmonic oscillator. We also do recover the appropriate values of $\tilde{\omega}_c$ and $\tilde{\omega}_t$ in the limit $\alpha \to m\omega_c$.

6. Show that the eigenstates of the lowest Landau level for $v_t \neq 0$ (current scenario) have the same form as without trapping potential (exercise 5) with the length scale l_B redefined to a new length scale. Determine this new length scale.

   Solution: By dimensional analysis, we always have

$$[\hat{\Pi}_x, \hat{\Pi}_y] = -i\hbar^2 l_B^{-2} \text{ and } [\hat{X}, \hat{Y}] = i l_B'^2. \tag{23}$$

   The length scale of both oscillators remains therefore equal here. Note again how dimensional analysis simplify all such computations. We therefore obtain:

$$\tilde{l}_B^{-4} = \frac{\alpha^2}{\hbar^2} = \frac{m v_t}{\hbar^2} + l_B^{-4} \tag{24}$$

   We can define the trapping length $l_t^4 = \frac{\hbar^2}{4 m v_t}$ to obtain the nice formula:

$$\tilde{l}_B^{-4} = l_t^{-4} + l_B^{-4}. \tag{25}$$

7. Give the corresponding eigenvalues in the lowest Landau level.

   Solution: From there, the rest of the mathematics is the same as before, with

$$\Phi_{0,m} = \frac{z^m}{\sqrt{2\pi \tilde{l}_B^{2m+2} m! 2^m}} e^{-\frac{zz^*}{4\tilde{l}_B^2}} \text{ and the energy } \hbar\left(\frac{\omega_c + \omega_t}{2} + m\omega_t\right) \tag{26}$$

8. Now, consider a system of N electrons in the trapping potential. For non-interacting electrons, under which condition can we construct the ground state without using states in the higher Landau levels?

   Solution: We want here to be able to say that all electrons prefer to be in the lowest Landau level rather than an orbital with lower angular momentum but in a higher Landau level. This means here that we want

$$\tilde{\omega}_c \gg \tilde{\omega}_t m_{\max}, \tag{27}$$

   where max is the largest occupied orbital. Neglecting edge effects, we therefore get

$$\tilde{\omega}_c \gg \tilde{\omega}_t N \nu^{-1}. \tag{28}$$

9. Write the wavefunction $\Psi_{1/3}(z_1, ..., z_N)$ of the Laughlin state at $\nu = \frac{1}{3}$ for the above case. What is its total angular momentum? What is its energy in the presence of the trapping potential?

   Solution: The Laughlin wave function is

$$\Psi_{1/m} = \prod_{i<j} (z_i - z_j)^m e^{-\sum_j \frac{|z_i|^2}{4\tilde{l}_B^2}}. \tag{29}$$

   At filling 1/3, we have $m = 3$. The total angular momentum correspond to the degree of the polynomial $\prod_{i<j}(z_i - z_j)^m$, i.e., $\frac{3N(N-1)}{2}$ (up to a sign, for simplification). Correspondingly, its energy in the trapping potential is

$$E_{1/3} = \hbar\frac{\tilde{\omega}_c}{2} + \hbar\tilde{\omega}_t / \frac{3N(N-1)}{2} + \frac{1}{2}). \tag{30}$$

10. What area does the state $\Psi_{1/3}$ approximately occupy?

Solution: We need to evaluate the area covered by a given orbital m in the lowest Landau level. We need to compute $\langle \Phi_{0,m}|r^2|\Phi_{0,m}\rangle$. A straightforward computation leads to:

$$\langle r^2 \rangle = \frac{1}{2\pi l_B^{2m+2} m! 2^m} \iint r^2 r^{2m} e^{-r^2/2/l_B^2} d^2\vec{r} \tag{31}$$

We can recognize here a computation that we have done before as the integral is exactly the one that we evaluate when normalizing $\Phi_{0,m+1}$, and therefore

$$\langle r^2 \rangle = \frac{2\pi l_B^{2m+2+2}(m+1)! 2^{m+1}}{2\pi l_B^{2m+2} m! 2^m} = 2(m+1) l_B^2. \tag{32}$$

The area covered by the circular orbital is therefore $2\pi(m+1)l_B^2$. For $\Phi_{1/3}$, the largest occupied orbital is $m_{\max} = 3(N-1)$, and therefore the state occupies an area

$$2\pi(3N-2)l_B^2. \tag{33}$$