Fractional quantum Hall effect: Laughlin wavefunction

1) Many derivations can also be found in Prof. Mila's lecture note.

(a) Anticipating the notations of the next question:

$$H_{\text{Landau}} = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m}\left(\hat{\Pi}_x^2 + \hat{\Pi}_y^2\right). \tag{1}$$

(b) The computation of the commutator is fairly standard:

$$[\hat{\Pi}_x, \hat{\Pi}_y] = [p_x, \frac{e}{c}A_y] - [p_y, \frac{e}{c}A_x] = -i\frac{e\hbar}{c}(\partial_x A_y - \partial_y A_x) = -i\frac{\hbar e B}{c}. \tag{2}$$

Now, we can define $\hat{P} = \hat{\Pi}_x$ and $\hat{Q} = \frac{c}{eB}\hat{\Pi}_y$. We then rewrite

$$H_{\text{Landau}} = \frac{1}{2m}\left(\hat{P}^2 + \frac{e^2B^2}{c^2}\hat{Q}^2\right) = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega_c^2\hat{Q}^2, \tag{3}$$

where

$$\omega_c = \frac{eB}{mc}. \tag{4}$$

(c) Working with P and Q as p_x and x, we recognize the Harmonic oscillator such that $[x, p_x] = i\hbar$. This directly gives us the results.

(d) We compute the two correlators

$$[\hat{X}, \hat{\Pi}_x] = [x, p_x] - \frac{1}{m\omega_c}[\hat{\Pi}_y, \hat{\Pi}_x] = i\hbar - i\hbar\frac{eB}{mc\omega_c} = 0 \tag{5}$$

$$[\hat{X}, \hat{\Pi}_y] = 0 \tag{6}$$

Similarly for $\hat{Y}$. Given the form of the Hamiltonian, we directly have $[\hat{X}, H_{\text{Landau}}] = 0 = [\hat{Y}, H_{\text{Landau}}]$. Just as an example:

$$[\hat{X}, \hat{\Pi}_y^2] = [\hat{X}, \hat{\Pi}_y]\hat{\Pi}_y + \hat{\Pi}_y[\hat{X}, \hat{\Pi}_y] = 0 \tag{7}$$

Finally,

$$[\hat{X}, \hat{Y}] = \frac{1}{m\omega_c}[x, p_x] - \frac{1}{m\omega_c}[p_y, y] - \frac{1}{m^2\omega_c^2}[\hat{\Pi}_x, \hat{\Pi}_y] = \frac{i\hbar}{m\omega_c} \tag{8}$$

(e) Here, we can prove the result in different minimal ways. The simplest is to note that we can formally rewrite the Hamiltonian as

$$H_{\text{Landau}} = \frac{1}{2m}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2) + 0 \times (\hat{X}^2 + \hat{Y}^2). \tag{9}$$

With the previously derived commutation relations, that means we can separately solve the two harmonic oscillators. Let b be the bosonic operator from the second oscillator. As a linear combination of $\hat{X}$ and $\hat{Y}$, it commutes with the Hamiltonian. The general eigenstates are

$$\Phi_{n,m} = \frac{(a^\dagger)^n(b^\dagger)^m}{\sqrt{n!m!}}|0\rangle \text{ with energy } \hbar(n + \frac{1}{2})\omega_c. \tag{10}$$

(f) This is where the gauge becomes relevant. Here, two ways offer to us: either we can start from the proposed equations and show that we recover the original Hamiltonian, or write the expression of a as a function of $\hat{\Pi}$. We remember that a must be linear combination of the Π and that

$$\hat{\Pi}_x^2 + \hat{\Pi}_y^2 = 2m\hbar\omega_c(a^\dagger a + \frac{1}{2}) \text{ and } [a, a^\dagger] = 1. \tag{11}$$

Let $a = \alpha_x \hat{\Pi}_x + \alpha_y \hat{\Pi}_y$. That means that

$$|\alpha_x|^2 = |\alpha_y|^2 = \frac{1}{2m\hbar\omega_c} \tag{12}$$

$$\alpha_x^* \alpha_y \hat{\Pi}_x \hat{\Pi}_y + \alpha_y^* \alpha_x \hat{\Pi}_y \hat{\Pi}_x = -\frac{1}{2} \tag{13}$$

$$(\alpha_y \alpha_x^* - \alpha_x \alpha_y^*) i\hbar m\omega_c = 1. \tag{14}$$

The first line is in fact not necessary to solve the problem. The second line imposes $\alpha_x^* \alpha_y$ to be purely imaginary, and the third line then fix the norm. Let $\alpha_x = \frac{i}{\sqrt{2\hbar m\omega_c}}$ and $\alpha_y = -i\alpha_x$. Then

$$a = \frac{1}{\sqrt{2\hbar m\omega_c}}(i\hat{\Pi}_x + \hat{\Pi}_y) = \frac{1}{\sqrt{2\hbar m\omega_c}}(i(-i\hbar\partial_x - \frac{eB}{c}\frac{y}{2}) + (-i\hbar\partial_y + \frac{eB}{c}\frac{x}{2}) \tag{15}$$

$$a = \frac{1}{\sqrt{2\hbar m\omega_c}}(\hbar(\partial_x - i\partial_y) + \frac{m\omega_c}{2}(x + iy)) = \sqrt{2}(\sqrt{\frac{\hbar}{m\omega_c}}\overline{\partial} + \sqrt{\frac{m\omega_c}{\hbar}}\frac{z}{4}). \tag{16}$$

We introduce the elementary length

$$l_B = \sqrt{\frac{\hbar}{m\omega_c}}, \tag{17}$$

which has the correct dimension, and obtain the expression of a we wanted. Note that l_B can be obtained also in the following way. We know that the energy scale is $\hbar\omega_c$, so we can factorize the second half of the Hamiltonian

$$\frac{1}{2}m\omega_c^2 Q^2 = \frac{\hbar\omega_c}{2}\frac{m\omega_c}{\hbar}Q^2. \tag{18}$$

Given that Q is a length, the natural length scale of the problem should be

$$l_B^2 = \frac{\hbar}{m\omega_c}. \tag{19}$$

The Hamiltonian can be rewritten in the suggestive form

$$H_{\text{Landau}} = \hbar\omega_c \left(\frac{l_B^2 \hat{P}^2}{2} + \frac{\hat{Q}^2}{2l_B^2} \right). \tag{20}$$

A similar computation gives the result for b. Note that we have a lot of freedom of choice in the definition of the orbitals. This form is the most convenient on the plane, and for identification with the angular momentum.

(g) We start with $n = m = 0$ The vacuum state verifies

$$a|0\rangle = b|0\rangle = 0, \tag{21}$$

which leads to

$$l_b \overline{\partial}\Phi_{0,0} + \frac{z}{4l_B}\Phi_{0,0} = 0 = l_b \partial\Phi_{0,0} + \frac{z^*}{4l_B}\Phi_{0,0}. \tag{22}$$

The first equation implies

$$\Phi_{0,0}(z,z^*) = e^{-\frac{zz^*}{4l_B^2}} f(z) \tag{23}$$

and the second

$$\Phi_{0,0}(z,z^*) = e^{-\frac{zz^*}{4l_B^2}} g(z^*) \tag{24}$$

Together, we have

$$\Phi_{0,0}(z,z^*) = Ce^{-\frac{zz^*}{4l_B^2}}. \tag{25}$$

The normalization constant is obtained by fixing the norm of $\Phi_{0,0}$ to 1

$$\iint d^2\vec{r}|\Phi_{0,0}|^2 = C^2 \iint d^2\vec{r} e^{-\frac{r^2}{2l_B^2}} = C^2 \times 2\pi l_B^2. \tag{26}$$

To obtain the general form, let s assume that it is correct up for $m \geq 0$.

$$|0, m+1\rangle = \frac{b^\dagger}{\sqrt{m+1}}|0,m\rangle \tag{27}$$

$$\begin{aligned}
\Phi_{0,m+1} &= \frac{\sqrt{2}}{\sqrt{m+1}}(-l_B\overline{\partial} + \frac{z}{4l_B})\Phi_{0,m} && (28)\\
&= \frac{\sqrt{2}}{\sqrt{m+1}}(-l_B(-\frac{z}{4l_B}) + \frac{z}{4l_B})\frac{z^m}{\sqrt{2\pi l_B^{2m+2} m! 2^m}} e^{-\frac{zz^*}{4l_B^2}} && (29)\\
&= \frac{z}{\sqrt{2l_B^2(m+1)}}\frac{z^m}{\sqrt{2\pi l_B^{2m+2} m! 2^m}} e^{-\frac{zz^*}{4l_B^2}}. && (30)
\end{aligned}$$

2) This part is mostly related to the Appendix p.77 in Prof. Mila's lecture note.

(a) The classical angular momentum is $L_z = xp_y - yp_x$.

(b)

$$\begin{aligned}
L_z &= xp_y - yp_x \\
&= (X + \frac{1}{m\omega_c}\Pi_y)(-\frac{1}{2}m\omega_c X + \frac{1}{2}\Pi_y) - (Y - \frac{1}{m\omega_c}\Pi_x)(\frac{1}{2}m\omega_c Y + \frac{1}{2}\Pi_x) \\
&= -\frac{1}{2}m\omega_c(X^2 + Y^2) + \frac{1}{2m\omega_c}(\Pi_x^2 + \Pi_y^2) \\
&= \hbar\left(a^\dagger a - b^\dagger b\right)
\end{aligned} \tag{31}$$

(c) This is completely in Prof. Mila's lecture note.

(d) Again, See the lecture note. We work in the lowest Landau level in this question, so we can throw away all the α operators, and just apply the results of the previous question for the operators $\beta_\pm$.

(e) We want a recurrence relation on n this time, so we use

$$|n+1, m\rangle = \frac{a^\dagger}{\sqrt{n+1}}|n,m\rangle \tag{32}$$

We obtain:

$$P_{n+1,m} = \frac{\sqrt{2}}{\sqrt{n+1}}(-\partial P_{n,m} + \frac{z^*}{2l_B}P_{n,m}) \tag{33}$$

By induction, if P is a polynomial at rank (n, m) it remains a polynomial at rank $(n+1, m)$. We also have $P_{0,m}$ a polynomial for all m, so $P_{n,m}$ is indeed a polynomial in z and z^*.

By construction, the angular momentum of $\Phi_{n,m}$ is $\hbar(n-m)$.

A computation similar to the previous exercice give us

$$\Phi_{n,0} = \frac{{z^*}^n}{\sqrt{2\pi l_B^{2n+2} n! 2^n}} e^{-\frac{zz^*}{4l_B^2}}. \tag{34}$$