(A.) The de Haas-van Alphen effect:

1. For a grand canonical ensemble with chemical potential μ, the partition function is defined as

$$Z = \sum_i e^{-\beta(\mathcal{E}_i - \mu N_i)}, \tag{1}$$

where $\beta = 1/(k_B T)$. Here, each microstate is labelled by i, and has total energy $\mathcal{E}_i$ and total particle number N_i. Let us a now consider a system of free fermions,

$$H = \sum_\alpha E_\alpha c_\alpha^\dagger c_\alpha, \tag{2}$$

Here α labels the different possible states in which each fermion can be with energy E_α. The microstates are described by the occupation number $n_\alpha \in \{0, 1\}$ of the different possible states α. We then have $\mathcal{E}_i = \sum_\alpha n_\alpha E_\alpha$ and $N_i = \sum_\alpha n_\alpha$. The partition function thus reads

$$\begin{aligned}
Z &= \sum_{n_{\alpha_1}=0}^{1} \sum_{n_{\alpha_2}=0}^{1} \sum_{n_{\alpha_3}=0}^{1} \cdots \left(\prod_\alpha e^{-\beta n_\alpha (E_\alpha - \mu)} \right) \\
&= \left(\sum_{n_{\alpha_1}=0}^{1} e^{-\beta n_{\alpha_1}(E_{\alpha_1} - \mu)} \right) \left(\sum_{n_{\alpha_2}=0}^{1} e^{-\beta n_{\alpha_2}(E_{\alpha_2} - \mu)} \right) \left(\sum_{n_{\alpha_3}=0}^{1} e^{-\beta n_{\alpha_3}(E_{\alpha_3} - \mu)} \right) \cdots \\
&= \prod_\alpha \sum_{n=0}^{1} e^{-\beta n (E_\alpha - \mu)} \\
&= \prod_\alpha \left(1 + e^{-\beta(E_\alpha - \mu)} \right).
\end{aligned} \tag{3}$$

In our system, we have

$$E_{n,k_z} = \hbar\omega_c n + \frac{\hbar^2 k_z^2}{2m} \qquad n = 0, 1, \ldots, \tag{4}$$

where each value with $n \neq 0$ occurs twice (due to the spin degeneracy including the Zeeman effect) and the value with $n = 0$ occurs only once. In addition, for a given k_z the degeneracy of each Landau level is (see course notes)

$$D = \frac{L_x L_y}{2\pi\hbar c} eB = \frac{L_x L_y \omega_c}{2\pi\hbar}. \tag{5}$$

The partition function is thus

$$Z = \prod_{k_z} \prod_{n=0}^{\infty} \left(1 + e^{-\beta(E_{n,k_z} - \mu)} \right)^{d(n)}, \tag{6}$$

where $d(n)$ is the number of states with energy E_{n,k_z}:

$$d(0) = D \qquad d(n) = 2D \quad \text{for } n \geq 1. \tag{7}$$

The free energy (or more precisely the grand potential, as we work in the grand canonical ensemble) reads

$$F = -\frac{1}{\beta}\log Z = -\frac{1}{\beta}\sum_{k_z}\sum_n d(n)\log\left(1+e^{-\beta(E_{n,k_z}-\mu)}\right) \tag{8}$$

In the limit where k_z takes continuous values, the expression in Eq. (6) is not so well behaved (it can still be dealt with). However, for the free energy we obtain

$$\begin{aligned} F &= -\frac{1}{\beta}\frac{L_z}{2\pi}\int_{-\infty}^{\infty} dk_z \sum_n d(n)\log\left(1+e^{-\beta(E_{n,k_z}-\mu)}\right) \\ &= -\frac{V\omega_c}{2\pi^2\hbar\beta}\int_{-\infty}^{\infty} dk_z \left[\frac{1}{2}\log\left(1+e^{-\beta\left(\frac{\hbar^2k_z^2}{2m}-\mu\right)}\right)+\sum_{n=1}^{\infty}\log\left(1+e^{-\beta\left(\hbar\omega_c n+\frac{\hbar^2k_z^2}{2m}-\mu\right)}\right)\right] \\ &= \hbar\omega_c\left[\frac{1}{2}f(\mu)+\sum_{n=1}^{\infty}f(\mu-\hbar\omega_c n)\right], \end{aligned} \tag{9}$$

where

$$f(\epsilon) = -\frac{mV}{2\pi^2\hbar^2\beta}\int_{-\infty}^{\infty} dk_z \log\left[1+e^{\beta\left(\epsilon-\frac{\hbar^2k_z^2}{2m}\right)}\right]. \tag{10}$$

Note that we use this notation so that the function f is independent of the magnetic field B.

2. By multiplying by $g(x)$ and by integrating x from 0 to ∞ on both sides of

$$\sum_{m=-\infty}^{\infty}\delta(x-m) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x}, \tag{11}$$

we obtain

$$\int_0^{\infty}\sum_{m=-\infty}^{\infty} g(x)\delta(x-m)dx = \int_0^{\infty} g(x)\delta(x)dx + \sum_{m=1}^{\infty}\int_0^{\infty} g(x)\delta(x-m)dx = \frac{1}{2}g(0)+\sum_{n=1}^{\infty}g(n), \tag{12}$$

and

$$\begin{aligned} \int_0^{\infty}\sum_{n=-\infty}^{\infty} g(x)e^{2\pi i n x}dx &= \int_0^{\infty} g(x)dx + \sum_{n=1}^{\infty}\int_0^{\infty} g(x)\left(e^{2\pi i n x}+e^{-2\pi i n x}\right)dx, \\ &= \int_0^{\infty} g(x)dx + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\int_0^{\infty} g(x)e^{2\pi i n x}dx, \end{aligned} \tag{13}$$

where we have used the fact that $g(x)$ is real. By equaling Eq. (12) to Eq. (13) we obtain the desired relation.

3. We can use the Poisson's formula for F with $g(x) = f(\mu-\hbar\omega_c x)$ and obtain

$$F = \hbar\omega_c\left[\int_0^{\infty} f(\mu-\hbar\omega_c x)dx + \sum_{n=1}^{\infty} 2\,\mathrm{Re}\int_0^{\infty} f(\mu-\hbar\omega_c x)e^{2\pi i n x}dx\right] = F_0+F_1. \tag{14}$$

We identify

$$F_0 = \hbar\omega_c\int_0^{\infty} dx f(\mu-\hbar\omega_c x), \quad \text{and} \quad F_1 = \frac{mV}{\beta\pi^2\hbar^2}\,\mathrm{Re}\sum_{n=1}^{\infty} I_n, \tag{15}$$

where

$$I_n = -\hbar\omega_c \int_0^\infty dx \int_{-\infty}^\infty dk_z \log\left[1 + e^{\beta\left(\mu - \hbar\omega_c x - \frac{\hbar^2 k_z^2}{2m}\right)}\right] e^{2\pi i n x}. \tag{16}$$

Using the change of variable $y = \hbar\omega_c x$ we have

$$F_0 = \hbar\omega_c \int_0^\infty \frac{dy}{\hbar\omega_c} f(\mu - y) = \int_0^\infty dy f(\mu - y), \tag{17}$$

which is independent of B as B is not involved in $f(\mu - y)$.

4. Let us use the change of variable $x \to \xi = \beta\Big(\hbar\omega_c x + \frac{\hbar^2 k_z^2}{2m} - \mu\Big)$:

$$I_n = -\frac{1}{\beta} \int_{-\infty}^\infty dk_z \int_{\beta\left(\frac{\hbar^2 k_z^2}{2m} - \mu\right)}^\infty d\xi \log\left(1 + e^{-\xi}\right) \exp\left(2\pi i n\left(\frac{\xi}{\beta\hbar\omega_c} - \frac{\hbar^2 k_z^2}{2m\hbar\omega_c} + \frac{\mu}{\hbar\omega_c}\right)\right). \tag{18}$$

Because we are in the limit $\hbar\omega_c \ll \mu$ and $1 \ll \beta\mu$, we first set the lower boundary of the ξ integral to $-\beta\mu$. Indeed, only small values of k_z contribute significantly to the k_z integral because of the oscillating factor $\exp\Big(2\pi i n \frac{\hbar^2 k_z^2}{2m\hbar\omega_c}\Big)$. Thus we have

$$\begin{aligned}
I_n &= -\frac{1}{\beta} e^{\frac{2\pi i n\mu}{\hbar\omega_c}} \int_{-\beta\mu}^\infty d\xi e^{\frac{2\pi i n}{\beta\hbar\omega_c}\xi} \log\left(1 + e^{-\xi}\right) \int_{-\infty}^\infty dk_z e^{-\frac{\pi i n\hbar^2}{m\hbar\omega_c} k_z^2} \\
&= -\frac{1}{\beta\hbar}\sqrt{\frac{m\hbar\omega_c}{n}} e^{\frac{2\pi i n\mu}{\hbar\omega_c} - i\frac{\pi}{4}} \int_{-\beta\mu}^\infty d\xi e^{\frac{2\pi i n}{\beta\hbar\omega_c}\xi} \log\left(1 + e^{-\xi}\right),
\end{aligned} \tag{19}$$

where we used the identity

$$\int_{-\infty}^\infty e^{-i\alpha k_z^2} dk_z = e^{-\frac{i\pi}{4}} \sqrt{\frac{\pi}{\alpha}}. \tag{20}$$

We now integrate by parts twice for the ξ integral :

$$I_n = \frac{1}{\beta\hbar}\sqrt{\frac{m\hbar\omega_c}{n}} e^{\frac{2\pi i n\mu}{\hbar\omega_c} - i\frac{\pi}{4}} \left(\frac{\beta\hbar\omega_c}{2\pi n}\right)^2 \int_{-\beta\mu}^\infty d\xi e^{\frac{2\pi i n}{\beta\hbar\omega_c}\xi} \frac{e^\xi}{(1 + e^\xi)^2} + \text{boundary terms}, \tag{21}$$

where we have used

$$\frac{d^2}{d\xi^2} \log\left(1 + e^{-\xi}\right) = \frac{e^\xi}{(1 + e^\xi)^2}, \tag{22}$$

which is a function that is finite around $\xi = 0$ and vanishes exponentially elsewhere. Because we are only interested in the $1/B$ oscillating behaviour of F, we can neglect the boundary terms in Eq. (21). Indeed, at the $\xi = \infty$ boundary the terms vanish, and at the $\xi = -\beta\mu$ boundary the oscillating factors exactly cancel. Finally, we set the lower boundary of the ξ integral to $-\infty$. This is a good approximation because $\beta\mu \gg 1$ and the dominating contribution to the ξ integral comes from around $\xi = 0$. We calculate

$$\int_{-\infty}^\infty d\xi e^{\frac{2\pi i n}{\beta\hbar\omega_c}\xi} \frac{e^\xi}{(1 + e^\xi)^2} = \frac{2\pi^2 n}{\beta\hbar\omega_c} \frac{1}{\sinh\left(\frac{2\pi^2 i n}{\hbar\omega_c\beta}\right)}. \tag{23}$$

Plugging the expression for I_n into Eq. (15), we get

$$F_1 = \frac{(m\hbar\omega_c)^{3/2} V}{2\pi^2\hbar^3\beta} \sum_{n=1}^\infty \frac{\cos\left(\frac{2\pi n\mu}{\hbar\omega_c} - \frac{\pi}{4}\right)}{n^{3/2}\sinh\left(\frac{2\pi^2 n}{\hbar\omega_c\beta}\right)} + \text{non-oscillating terms}. \tag{24}$$

5. Using
$$\frac{d}{dB}\cos\left(\frac{2\pi n\mu}{\hbar\omega_c}-\frac{\pi}{4}\right)=\frac{2\pi n\mu}{\hbar\omega_c}\frac{1}{B}\sin\left(\frac{2\pi n\mu}{\hbar\omega_c}-\frac{\pi}{4}\right), \tag{25}$$
and assuming that all the other terms are independent of B, we obtain
$$\frac{1}{B}\frac{\partial}{\partial B}\left(\frac{F}{V}\right)=-\frac{m^{3/2}\mu\sqrt{\hbar\omega_c}}{B^2\pi\hbar^3\beta}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2\pi n\mu}{\hbar\omega_c}-\frac{\pi}{4}\right)}{\sqrt{n}\sinh\left(\frac{2\pi^2 n}{\hbar\omega_c\beta}\right)}. \tag{26}$$

6. The oscillating parts of the susceptibility, when plotted against $1/B$, have frequencies of $f=\frac{n\mu mc}{\hbar e}$ with $n=1,2,\dots$, and thus the overall period is
$$P=\frac{\hbar e}{\mu mc}, \tag{27}$$
corresponding to the $n=1$ term. Note that in the strong field limit $k_BT\lesssim\hbar\omega_c$, the higher harmonics (terms with $n>1$) are negligible because of the $\sinh\big(2\pi^2 n/(\hbar\omega_c\beta)\big)$ factor. Indeed, for $\hbar\omega_c\beta=1$ we have $\sinh\big(4\pi^2\big)/\sinh\big(2\pi^2\big)\sim 10^8$.

The following explains the significance of the de Haas-van Alphen effect:

Let us consider a plane perpendicular to k_z in k-space. It intersects the Fermi surface with a certain cross-section, which depends on the position of the plane. Let us denote A_e the extremum of this cross-section. For free electrons, we have a spherical Fermi surface with radius $k_F=\sqrt{2m\mu}/\hbar$, and a maximum of the cross-section for the plane at $k_z=0$ so that $A_e=\pi k_F^2$. Rewriting the period in terms of A_e we obtain
$$P=\frac{2\pi e}{\hbar c}\frac{1}{A_e}. \tag{28}$$
For a more general Fermi surface, there is a singularity in the density of state at the Fermi energy every time a Landau level crosses the Fermi energy. Remarkably, Eq. (28) still holds in the general case. For a more complicated Fermi surface, there can be several different extremal cross-sections corresponding to different periods. The cross-sections may also change for different orientations of B.

Thus, extremal Fermi surface areas can be detected through the de Haas-van Alphen effect.

7. In the $\hbar\omega_c\ll k_BT$ limit, $1/(\beta\hbar\omega_c)\gg 1$, and thus
$$\sinh\left(\frac{2\pi^2 n}{\hbar\omega_c\beta}\right)\approx\frac{1}{2}e^{\frac{2\pi^2 n}{\hbar\omega_c\beta}}, \tag{29}$$
so that the amplitude of the oscillation decays as $e^{-\frac{2\pi^2}{\hbar\omega_c\beta}}$.