

In this series, we want to study the electron-pairing mechanism under the effect of an attractive two-body interaction $V(\mathbf{r}_1, \mathbf{r}_2)$,

$$H(\mathbf{r}_1, \mathbf{r}_2) = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) + V(|\mathbf{r}_1 - \mathbf{r}_2|). \quad (1)$$

This formulation is called the Cooper problem (L. N. Cooper, Phys. Rev. **104**, 1189 (1956)). To this end, we consider the wave function of vanishing total momentum $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = 0$:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \Psi(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{\mathbf{k}} \Psi_{\mathbf{k}} e^{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)}. \quad (2)$$

In addition, we assume that Ψ is even, $\Psi(\mathbf{r}) = \Psi(-\mathbf{r})$, which implies that $\Psi_{\mathbf{k}} = \Psi_{-\mathbf{k}}$. As the two-electron state must be antisymmetric, this choice implies that the total spin must be zero ($S = 0$).

1. By setting

$$\epsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m \quad \text{and} \quad V_{\mathbf{k}-\mathbf{k}'} = \int d^3\mathbf{r} V(\mathbf{r}) e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}},$$

write the Schrödinger equation $H\Psi = E\Psi$ in the momentum representation. It is recalled that

$$\frac{1}{\Omega} \int d^3\mathbf{r} e^{i\mathbf{k} \cdot \mathbf{r}} = \delta_{\mathbf{k}, \mathbf{0}}.$$

2. Assuming that the interaction does not depend on the momentum and is attractive, that is, $V_{\mathbf{k}-\mathbf{k}'} = -|v|$, show that the dispersion relation can be written in the form

$$\frac{1}{\Omega} \sum_{\mathbf{k}'} \frac{1}{E - 2\epsilon_{\mathbf{k}'}} = -\frac{1}{|v|}. \quad (3)$$

3. We now assume that

- both electrons are in the presence of a Fermi sea,
- the attractive interaction comes from the effective electron-electron interaction due to phonons (see course). For simplicity, we assume that the interaction is attractive and independent of momentum ($-|v|$) if $\epsilon_F \leq \epsilon_{\mathbf{k}}, \epsilon_{\mathbf{k}'} \leq \epsilon_F + \hbar\omega_D$ and that it vanishes otherwise. $\hbar\omega_D$ is a characteristic energy of the phonons, which corresponds in general to the Debye frequency ω_D .

Writing the energy of the two electrons as $E = 2\epsilon_F - \epsilon_b$, where $2\epsilon_F$ is the energy of the electrons without interaction and $\epsilon_b \geq 0$ is the binding energy of the electrons, calculate the binding energy ϵ_b for a weak interaction (in the sense that $\epsilon_b \ll \epsilon_F$) and with $\hbar\omega_D \ll \epsilon_F$ (always true for metals).

It is recalled that for $x > b > 0$,

$$\int dx \frac{x^2}{(x^2 - b^2)} = x + \frac{b}{2} \ln \left| \frac{x - b}{x + b} \right|. \quad (4)$$

4. Using the density of states per spin at the Fermi level $\rho_F = mk_F/(2\pi^2\hbar^2)$, show that in the $\rho_F|v| \ll 1$ limit,

$$\epsilon_b \simeq 2(\hbar\omega_D)e^{-\frac{2}{\rho_F|v|}} \quad (5)$$

and that in the $\rho_F|v| \gg 1$ limit (strong coupling)

$$\epsilon_b \simeq (\hbar\omega_D)\rho_F|v|. \quad (6)$$