Fractional quantum Hall effect in a trapping potential:

In this exercise, we will investigate the effect of a simple trapping potential $V_t(r) = \frac{1}{2}v_t r^2$ on the Laughlin states. We consider spinless electrons in two dimensions.

[This exercise is a continuation of Exercise 5 and significantly dependent on Exercise 5 where we discussed the case without a trapping potential. The students need to frequently refer to the that while proceeding with this exercise.]

In exercise 5, we have introduced the operators

$$\hat{\Pi}_{x/y} = \hat{p}_{x/y} + \frac{e}{c}A_{x/y}. \tag{1}$$

1. Now, we define the following operators

$$\hat{\Pi}_x(\alpha) = \hat{p}_x - \frac{\alpha}{2}y \tag{2}$$
$$\hat{\Pi}_y(\alpha) = \hat{p}_y + \frac{\alpha}{2}x \tag{3}$$
$$\hat{X}(\alpha) = \hat{x} - \alpha^{-1}\hat{\Pi}_y(\alpha) \tag{4}$$
$$\hat{Y}(\alpha) = \hat{y} + \alpha^{-1}\hat{\Pi}_x(\alpha). \tag{5}$$

   Express $\hat{\Pi}_x^2$ as a function of $\hat{\Pi}_x^2(\alpha)$, $\hat{Y}(\alpha)^2$ and y^2.

   [Hint: You need to use the commutation relations derived in Exercise 5.]

2. In the similar fashion, express $\hat{\Pi}_y^2$ as a function of $\hat{\Pi}_y^2(\alpha)$, $\hat{X}(\alpha)^2$ and x^2.

3. Show that by choosing α appropriately, one can rewrite the full Hamiltonian, i.e.

$$H_{\text{Landau}} + V_t(r) = \frac{1}{2m}(\hat{\Pi}_x^2 + \hat{\Pi}_y^2) + \frac{1}{2}v_t r^2,$$

   in the form

$$\lambda_1\hat{\Pi}_x(\alpha)^2 + \lambda_2\hat{\Pi}_y(\alpha)^2 + \lambda_3\hat{X}(\alpha)^2 + \lambda_4\hat{Y}(\alpha)^2. \tag{6}$$

   Find the expressions of the λ_j's.

4. First, convince yourself that the above Hamiltonian is made of two independent harmonic oscillators and thus, can be written as:

$$H_{\text{Landau}} + V_1 = \hbar\tilde{\omega}_c(a^\dagger a + \frac{1}{2}) + \hbar\omega_t(b^\dagger b + \frac{1}{2}) \tag{7}$$

   Now, determine the frequencies $\tilde{\omega}_c$ and ω_t.

   [Hint: You can use an analogy from Exercise 5.]

5. What is ω_t at $v_t = 0$? What does it signify?

6. Show that the eigenstates of the lowest Landau level for $v_t \neq 0$ (current scenario) have the same form as without trapping potential (exercise 5) with the length scale l_B redefined to a new length scale. Determine this new length scale.

7. Give the corresponding eigenvalues in the lowest Landau level.
8. Now, consider a system of N electrons in the trapping potential. For non-interacting electrons, under which condition can we construct the ground state without using states in the higher Landau levels?
9. Write the wavefunction $\Psi_{1/3}(z_1, ..., z_N)$ of the Laughlin state at $\nu = \frac{1}{3}$ for the above case. What is its total angular momentum? What is its energy in the presence of the trapping potential?
10. What area does the state $\Psi_{1/3}$ approximately occupy?