**Fractional quantum Hall effect: Laughlin wavefunction** In this exercise, we will try to derive the Laughlin wavefunction without trapping potential step by step starting from the Hamiltonian, and look at its properties in detail, as it plays an important role in QHE. We consider spinless electrons in **two** dimensions.

**1)** In this question, we rederive the expression of the single-particle orbitals in the symmetric gauge. NB: Questions (a) to (e) can be done without specifying the gauge.

(a) Write the general expression of the Hamiltonian of a particule of charge $-e$ in a uniform magneric field.

(b) We introduce the operators

$$\hat{\Pi}_{x/y} = \hat{p}_{x/y} + \frac{e}{c}A_{x/y}. \tag{1}$$

Compute the commutator $[\hat{\Pi}_x, \hat{\Pi}_y]$. Show that we can write the Hamiltonian as

$$H_{\text{Landau}} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega_c^2\hat{Q}^2 \tag{2}$$

with $[\hat{Q}, \hat{P}] = i\hbar$.

(c) Show that the Hamiltonian can be rewritten as

$$H = \hbar\omega_c(a^\dagger a + \frac{1}{2}) \tag{3}$$

with $a, a^\dagger$ bosonic operators verifying $[a, a^\dagger] = 1$.

(d) We now introduce the operators

$$\hat{X} = \hat{x} - \frac{1}{m\omega_c}\hat{\Pi}_y \tag{4}$$

$$\hat{Y} = \hat{y} + \frac{1}{m\omega_c}\hat{\Pi}_x. \tag{5}$$

Show that they commute with the Hamiltonian and compute the commutator $[\hat{X}, \hat{Y}]$.

(e) Show that there exist $b, b^\dagger$ bosonic operators such that $[b, H_{\text{Landau}}] = [b^\dagger, H_{\text{Landau}}] = 0$. Deduce the general form of the eigenstates of $H_{\text{Landau}}$ in terms of $a$ and $b$, and give the corresponding energies.

(f) We use the convention $z = x - iy$ and $\partial = (\partial_x + i\partial_y)/2$. Show that, in the symmetric gauge defined by

$$A_x = -\frac{By}{2}, \qquad A_y = \frac{Bx}{2}, \tag{6}$$

one can rewrite the $a$ and $b$ operators such that:

$$a = \sqrt{2}\left(\frac{z}{4l_B} + l_B\bar{\partial}\right), \tag{7}$$

$$b = \sqrt{2}\left(\frac{z^*}{4l_B} + l_B\partial\right), \tag{8}$$

where $l_B$ is a length scale to be determined.

(g) Show that the eigenstates of $H_{\text{Landau}}$ in the lowest Landau level can be written as

$$\Phi_{n=0,m} = \frac{1}{\sqrt{2\pi l_B^2 m!}}\left(\frac{z}{\sqrt{2}l_B}\right)^m e^{-|z|^2/4l_B^2}. \tag{9}$$

Note that if we write $z = re^{-i\phi}$, we reach exactly the same expression as Eq.(4.43) in Sigrist's textbook.

Hint: start from the state with $n = m = 0$ and use induction.

**2)** In this question, we discuss the orbital momentum of the electrons.

(a) What is the angular momentum of a classical particle?

(b) Using that definition, express $\hat{L}^z$ as a function of $\hat{\Pi}_x$, $\hat{\Pi}_y$, $\hat{X}$ and $\hat{Y}$. Deduce its expression in terms of $a$ and $b$.

(c) We introduce the center of mass and the relative coordinates for two electrons:

$$\begin{aligned} \vec{R}_{\text{CM}} &= \frac{\vec{r}_1 + \vec{r}_2}{2} \text{ and } \vec{r}_{\text{rel}} = \vec{r}_1 - \vec{r}_2, \\ \vec{P}_{\text{CM}} &= \vec{p}_1 + \vec{p}_2 \text{ and } \vec{p}_{\text{rel}} = \frac{\vec{r}_1 - \vec{r}_2}{2}. \end{aligned} \tag{10}$$

What is the relation between $\vec{L}_1$, $\vec{L}_2$, $\vec{L}_{\text{CM}}$ and $\vec{L}_{\text{rel}}$? Show that one can rewrite

$$\hat{L}^z_{\text{CM}} = \hbar(\alpha_+^\dagger\alpha_+ - \beta_+^\dagger\beta_+), \qquad \hat{L}^z_{\text{rel}} = \hbar(\alpha_-^\dagger\alpha_- - \beta_-^\dagger\beta_-) \tag{11}$$

with

$$\sqrt{2}\alpha_\pm = a_1 \pm a_2, \qquad \sqrt{2}\beta_\pm = b_1 \pm b_2. \tag{12}$$

(d) Show that $\hat{L}^z_{\text{CM}}$ and $\hat{L}^z_{\text{rel}}$ commute. Let's now work in the lowest Landau levels, and let $|M, m\rangle$ the two particle state defined by

$$\hat{L}^z_{\text{CM}}|M, m\rangle = \hbar M|M, m\rangle, \tag{13}$$

$$\hat{L}^z_{\text{rel}}|M, m\rangle = \hbar m|M, m\rangle. \tag{14}$$

Show that it is proportional to

$$(z_1 + z_2)^M (z_1 - z_2)^m e^{-\sum_i |z_i|^2/4l_B^2} \tag{15}$$

(e) Turn back to the one-particle wavefunction. We note

$$\Phi_{n,m} = P_{n,m}(z, z^*)e^{-|z|^2/4l_B^2}. \tag{16}$$

Derive the recurrence equation on $n$ satisfied by $P_{n,m}$ and deduce that $P_{n,m}$ is a polynomial in $z$ and $z^*$. What is the angular momentum of the orbital $\Phi_{n,m}$? Finally, give the form of $\Phi_{n,0}$.