(A.) The de Haas-van Alphen effect: The de Haas-van Alphen effect is an oscillatory variation of the diamagnetic susceptibility as a function of the magnetic field strength (B). The method provides details of the extremal areas of a Fermi surface. In 1930, de Haas and van Alphen measured the magnetization M of the semimetal bismuth (Bi) as a function of B. They observed that the magnetic susceptibility M/B is a periodic function of $1/B$. This phenomenon is typically observed at low temperatures and high magnetic fields in metals that satisfy $k_BT \lesssim \hbar\omega_c \ll \mu$, where μ is the chemical potential.

The energy of free electrons in a strong magnetic field is given by

$$E_{n,k_z,\pm} = \hbar\omega_c(n+\frac{1}{2}) + \frac{\hbar^2 k_z^2}{2m} \pm \frac{1}{2} g\mu_B B \qquad n = 0, 1, 2, \ldots,$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. The last term is the coupling between the spin of the electrons and the magnetic field (Zeeman effect), $g = 2$ is the Landé factor and $\mu_B = \frac{\hbar e}{2mc}$ is the Bohr magneton. The terms can be rearranged as

$$E_{n,k_z} = \hbar\omega_c n + \frac{\hbar^2 k_z^2}{2m} \qquad n = 0, 1, \ldots,$$

where each value with $n \neq 0$ occurs twice and the value with $n = 0$ occurs only once (and do not forget the Landau level degeneracy[1]).

1. Using the definition of the free energy, $F = -k_BT \log Z$, where Z is the grand partition function, show that (*hint: First derive the expression of F for a general fermionic system with energy spectrum $\{E_\alpha\}$, and then replace $\{E_\alpha\}$ by $\{E_{n,k_z}\}$ with the degeneracy*).

$$F = \hbar\omega_c\left[\frac{1}{2}f(\mu) + \sum_{n=1}^{\infty} f(\mu - \hbar\omega_c n)\right],$$

   where

$$f(\epsilon) = -\frac{mV}{2\pi^2\hbar^2\beta}\int_{-\infty}^{\infty} dk_z \log\left[1 + e^{\beta\left(\epsilon - \frac{\hbar^2 k_z^2}{2m}\right)}\right] \tag{1}$$

2. Derive the Poissons's formula

$$\frac{1}{2}g(0) + \sum_{n=1}^{\infty} g(n) = \int_0^{\infty} g(x)dx + \sum_{n=1}^{\infty} 2\mathrm{Re}\int_0^{\infty} g(x)e^{2\pi i n x}dx \tag{2}$$

   for a general function $g(x)$. In order to do so, use the following Fourier series

$$\sum_{m=-\infty}^{\infty} \delta(x-m) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x},$$

   and $\int_0^{\infty} \delta(x)g(x)dx = \frac{1}{2}g(0)$. Here $\delta(x)$ is the Dirac delta function.

[1] We also recommend reading Section 3.1 of the lecture notes by Prof.Frédéric Mila for the explanation of Landau levels.

3. Using the Eq. (2) with $g(x) = f(\mu - \hbar\omega_c x)$ defined in Eq. (1), show that the free energy can be rewritten as

$$F = F_0 + F_1, \qquad F_0 = \hbar\omega_c \int_0^\infty dx f(\mu - \hbar\omega_c x), \qquad F_1 = \frac{mV}{\beta\pi^2\hbar^2} \mathrm{Re} \sum_{n=1}^{\infty} I_n.$$

   Show that F_0 is independent of B (without explicitly doing the integral). What is the expression of I_n?

4. By integrating by parts twice, show that in the limit $k_B T \ll \mu$,

$$F_1 = \frac{(m\hbar\omega_c)^{3/2} V}{2\pi^2\hbar^3\beta} \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi n\mu}{\hbar\omega_c} - \frac{\pi}{4}\right)}{n^{3/2} \sinh\left(\frac{2\pi^2 n}{\hbar\omega_c \beta}\right)}. \tag{3}$$

   In order to do so, you may need the change of variable $x \to \xi = \beta(\hbar\omega_c x + \frac{\hbar^2 k_z^2}{2m} - \mu)$ and the following integrals:

$$\int_{-\infty}^{\infty} e^{-i\alpha k_z^2} dk_z = e^{-\frac{i\pi}{4}} \sqrt{\frac{\pi}{\alpha}},$$

$$\int_{-\infty}^{\infty} \frac{e^{\xi}}{(e^{\xi}+1)^2} e^{i\alpha\xi} d\xi = \frac{\pi\alpha}{\sinh(\pi\alpha)}.$$

   (*hint: The fact that 1. $k_B T \ll \mu$; 2. only small values of k_z contribute significantly to the k_z integral because of the oscillating factor; 3. the dominating contribution to the ξ integral comes from around $\xi = 0$ means you can adjust the lower boundary of the ξ integral safely.*)

5. Calculate the magnetic susceptibility

$$\frac{M}{B} = -\frac{1}{B}\frac{\partial}{\partial B}\left(\frac{F}{V}\right) \approx -\frac{m^{3/2}\mu\sqrt{\hbar\omega_c}}{B^2\pi\hbar^3\beta} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2\pi n\mu}{\hbar\omega_c} - \frac{\pi}{4}\right)}{\sqrt{n}\sinh\left(\frac{2\pi^2 n}{\hbar\omega_c\beta}\right)},$$

   where M is the magnetisation density. We assume that only the most rapidly oscillating factors (the cosine term in Eq. (3)) needs to be differentiated.

6. What is the period of the oscillation?

7. Show that in the limit $\hbar\omega_c \ll k_B T$ (small magnetic field), the amplitude of the oscillation vanishes exponentially with $k_B T/(\hbar\omega_c)$.