Particle Physics 1 : Exercise 6

1) Dirac and Klein-Gordon equations

By operating on the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

with $\gamma^\nu \partial_\nu$, prove that the components of ψ satisfy the Klein-Gordon equation.

2) Spin and angular momentum of a non-relativistic free particle

Show that

(a) $[\hat{\mathbf{p}}^{\mathbf{2}}, \hat{\mathbf{r}} \times \hat{\mathbf{p}}] = 0$

(b) $[\hat{\mathbf{p}}^{\mathbf{2}}, \hat{\mathbf{S}}] = 0$

where the spin operator is given by $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\sigma}$ for a 2-dimensional spinor.
Note that the above relations imply that the Hamiltonian of a non-relativistic free particle commutes with both the angular momentum and the spin operators.

3) Spin and angular momentum of a relativistic Dirac particle

(a) Using $\hat{\mathrm{H}}_{\mathrm{D}} = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m$, show that

  (1) $[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{\mathbf{L}}] = -i\,\boldsymbol{\alpha} \times \hat{\mathbf{p}}$ and

  (2) $[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{\mathbf{S}}] = i\,\boldsymbol{\alpha} \times \hat{\mathbf{p}}$, where $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\Sigma}$ for a 4-components Dirac spinor,

  i.e. the sum of the angular momentum and spin operators, $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, commutes with $\hat{\mathrm{H}}_{\mathrm{D}}$, but $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ individually do not.

(b) Prove that the spinors of particles moving along the z direction are eigenfunctions with eigenvalues of $\pm\frac{1}{2}$ of the operator corresponding to the third component of the spin : $\hat{\mathrm{S}}_{\mathrm{z}} = \frac{1}{2}\Sigma_z$.

4) Helicity

Explain what helicity is and verify that the helicity operator

$$h = \frac{\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}}{2\mathrm{p}} = \frac{1}{2\mathrm{p}} \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix}$$

commutes with the Dirac Hamiltonian

$$\hat{\mathrm{H}}_{\mathrm{D}} = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m.$$

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