Particle Physics 1 : Exercise 6

1) Dirac and Klein-Gordon equations

Acting on the Dirac equation with $\gamma^\nu \partial_\nu$ and multiplying by $-i$ gives

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi + mi\gamma^\nu \partial_\nu \psi = 0.$$

Since ψ satisfies the Dirac equation, $i\gamma^\nu \partial_\nu \psi = m\psi$, and thus

$$(\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\psi = 0.$$

Since the order of differentiation doesn't matter $\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu$ can be written as

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu = \frac{1}{2}(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu)\partial_\nu \partial_\mu = g^{\mu\nu}\partial_\nu \partial_\mu,$$

and therefore

$$\begin{aligned} &(g^{\mu\nu}\partial_\nu \partial_\mu + m^2)\psi = 0 \\ \Rightarrow\ &(\partial^\mu \partial_\mu + m^2)\psi = 0 \end{aligned}$$

which is the Klein-Gordon equation.

2) Spin and angular momentum of a non-relativistic free particle

(a) Consider the commutator of $\hat{\mathbf{p}}^2$ with the x-component of $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y.$$

$$\begin{aligned} \left[\hat{\mathbf{p}}^2, \hat{L}_x\right] &= \left[\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2,\, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right] \\ &= \left[\hat{p}_y^2, \hat{y}\hat{p}_z\right] - \left[\hat{p}_z^2, \hat{z}\hat{p}_y\right] \\ &= \left[\hat{p}_y^2, \hat{y}\right]\hat{p}_z - \left[\hat{p}_z^2, \hat{z}\right]\hat{p}_y, \end{aligned} \tag{1}$$

where the last line follows from the fact that $\hat{p}_z$ commutes with $\hat{y}$ and $\hat{p}_y$ commutes with $\hat{z}$. Using

$$[\hat{y}, \hat{p}_y] = \hat{y}\hat{p}_y - \hat{p}_y\hat{y} = i$$

the commutators in the previous expression can be simplified as, for example :

$$\begin{aligned} \left[\hat{p}_y^2, \hat{y}\right] &= \hat{p}_y\hat{p}_y\hat{y} - \hat{y}\hat{p}_y\hat{p}_y \\ &= \hat{p}_y(\hat{y}\hat{p}_y - i) - \hat{y}\hat{p}_y\hat{p}_y \\ &= \hat{p}_y\hat{y}\hat{p}_y - i\hat{p}_y - \hat{y}\hat{p}_y\hat{p}_y \\ &= (\hat{y}\hat{p}_y - i)\hat{p}_y - i\hat{p}_y - \hat{y}\hat{p}_y\hat{p}_y \\ &= -2i\hat{p}_y. \end{aligned}$$

Similarly, $[\hat{p}_z^2, \hat{z}] = -2i\hat{p}_z$, and therefore Equation (1) becomes

$$\left[\hat{\mathbf{p}}^2, \hat{L}_x\right] = \left[\hat{p}_y^2, \hat{y}\right]\hat{p}_z - \left[\hat{p}_z^2, \hat{z}\right]\hat{p}_y = -2i\hat{p}_y\hat{p}_z + 2i\hat{p}_z\hat{p}_y = 0.$$

(b) Since the Pauli spin matrices do not depend on the spatial coordinates, it is fairly straightforward to see that all three components of the spin operator commute with $\hat{\mathbf{p}}^2$.

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3) Spin and angular momentum of a relativistic Dirac particle

(a) As in the previous exercise, we will compute the commutator between the Dirac Hamitonian $\hat{\mathrm{H}}_{\mathrm{D}}$ and the first component of the angular momentum $\hat{L}_x$:

$$\begin{aligned}\left[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{L}_x\right] &= [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} + \beta m, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y] \\ &= [\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y] \\ &= [\alpha_x\hat{p}_x + \alpha_y\hat{p}_y + \alpha_z\hat{p}_z, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y]\end{aligned}$$

Given the commutator between $\hat{x}_j$ and $\hat{p}_k$,

$$[x_j, p_k] = \delta_{jk} i, \tag{2}$$

the above expression can be simplified as

$$\begin{aligned}\left[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{L}_x\right] &= [\alpha_y\hat{p}_y + \alpha_z\hat{p}_z, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y] \\ &= [\alpha_y\hat{p}_y, \hat{y}\hat{p}_z] - [\alpha_z\hat{p}_z, \hat{z}\hat{p}_y] \\ &= \alpha_y\hat{p}_z\,[\hat{p}_y, \hat{y}] - \alpha_z\hat{p}_y\,[\hat{p}_z, \hat{z}]\,,\end{aligned} \tag{3}$$

where the last equality follows from the fact that $\hat{p}_y$ and $\hat{p}_z$ commute. Using again Equation (2), the commutator in Equation (3) can be finally written as

$$\left[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{L}_x\right] = -i\alpha_y\hat{p}_z + i\alpha_z\hat{p}_y = -i(\alpha_y\hat{p}_z - \alpha_z\hat{p}_y) = -i(\boldsymbol{\alpha}\times\hat{\mathbf{p}})_x.$$

After repeating a similar computation for $\hat{L}_y$ and $\hat{L}_z$, the final result for the angulat momentum is

$$\left[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{\mathbf{L}}\right] = -i(\boldsymbol{\alpha}\times\hat{\mathbf{p}}).$$

The spin operator acting on a 4-component Dirac spinor is

$$\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}\end{pmatrix}.$$

In order to compute the commutator between the Dirac Hamiltonian and the three component of $\hat{\mathbf{S}}$, one needs to derive the comutators between the Σ matrices and the matrices appearing in $\hat{\mathrm{H}}_{\mathrm{D}}$, $\boldsymbol{\alpha}$ and β, which in the Dirac-Pauli representation correspond to

$$\alpha_j = \begin{pmatrix}0 & \sigma_j \\ \sigma_j & 0\end{pmatrix} \text{ and } \beta = \begin{pmatrix}\mathbf{1} & 0 \\ 0 & -\mathbf{1}\end{pmatrix}.$$

It is straightforward to show that $[\beta, \Sigma_k]$. Then

$$\begin{aligned}[\alpha_j, \Sigma_k] &= \begin{pmatrix}0 & \sigma_j \\ \sigma_j & 0\end{pmatrix}\cdot\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix} - \begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}\cdot\begin{pmatrix}0 & \sigma_j \\ \sigma_j & 0\end{pmatrix} \\ &= \begin{pmatrix}0 & \sigma_j\sigma_k \\ \sigma_j\sigma_k & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma_k\sigma_j \\ \sigma_k\sigma_j & 0\end{pmatrix} = \begin{pmatrix}0 & [\sigma_j, \sigma_k] \\ [\sigma_j, \sigma_k] & 0\end{pmatrix} \\ &= 2i\epsilon_{jkl}\begin{pmatrix}0 & \sigma_l \\ \sigma_l & 0\end{pmatrix} = 2i\epsilon_{jkl}\,\alpha_l\end{aligned} \tag{4}$$

where the last step follows from the commutation relation of the Pauli spin matrices $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\,\sigma_l$. Given the commutator in Equation 4, one can easily compute the commutator between the Dirac Hamiltonian and the first component of the Spin :

$$\begin{aligned}\left[\hat{\mathrm{H}}_{\mathrm{D}}, \hat{S}_x\right] &= \frac{1}{2}\left[\alpha_y \hat{p}_y + \alpha_z \hat{p}_z, \Sigma_x\right] \\ &= \frac{1}{2}\hat{p}_y\left[\alpha_y, \Sigma_x\right] + \frac{1}{2}\hat{p}_z\left[\alpha_z, \Sigma_x\right] \\ &= \frac{1}{2}\hat{p}_y 2i\epsilon_{yxz}\alpha_z + \frac{1}{2}\hat{p}_z 2i\epsilon_{zxy}\alpha_y \\ &= -i\alpha_z\hat{p}_y + i\alpha_y\hat{p}_z \\ &= i(\alpha_y\hat{p}_z - \alpha_z\hat{p}_y) = i(\boldsymbol{\alpha}\times\hat{\mathbf{p}})_x.\end{aligned}$$

A similar computation can be carried out for $\hat{S}_y$ and $\hat{S}_z$ to show that

$$\left[\hat{\mathrm{H}}_{\mathrm{D}}, S_x\right] = i(\boldsymbol{\alpha}\times\hat{\mathbf{p}}).$$

It is worth noting that in non-relativistic quantum mechanics (previous exercise), both spin and angular momentum commute with the free-particle Hamiltonian, while in relativistic quantum mechanics only the sum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ does. This implies that, in general, it is not possible to identify a complete set of Dirac spinor states that are simultaneously eigentsates of $\hat{\mathrm{H}}_{\mathrm{D}}$ and e.g. $\hat{S}_z$, as was instead the case for 2-component spinors in non-relativistic quantum mechanics. However, in the specific case where the particle momentum is aligned with the z axis, the solutions to the Dirac equation (which are eigenstates of $\hat{\mathrm{H}}_{\mathrm{D}}$) are also eigenstates of $\hat{S}_z$ with eigenvalues $\pm\frac{1}{2}$, as proven in the second part of the exercise.

(b) If the momentum $\mathbf{p}$ is parallel to the z axis, the four particle solutions to the Dirac equation reduce to

$$u_1 = \sqrt{E+m}\begin{pmatrix}1\\0\\\frac{\mathrm{p}}{E+m}\\0\end{pmatrix} \qquad u_2 = \sqrt{E+m}\begin{pmatrix}0\\1\\0\\\frac{-\mathrm{p}}{E+m}\end{pmatrix}$$

$$u_3 = \sqrt{E+m}\begin{pmatrix}\frac{\mathrm{p}}{E-m}\\0\\1\\0\end{pmatrix} \qquad u_4 = \sqrt{E+m}\begin{pmatrix}0\\\frac{-\mathrm{p}}{E-m}\\0\\1\end{pmatrix}$$

where u_1 and u_2 are the positive energy solutions with $E = \sqrt{\mathbf{p}^2 + m^2}$, while u_3 and u_4 are the negative energy solutions with $E = -\sqrt{\mathbf{p}^2 + m^2}$. Given the operator for the third component of the spin

$$\hat{\mathrm{S}}_{\mathrm{z}} = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},$$

it is straightforward to show that

$$\begin{aligned}\hat{\mathrm{S}}_{\mathrm{z}}u_1 &= \frac{1}{2}u_1 \\ \hat{\mathrm{S}}_{\mathrm{z}}u_2 &= -\frac{1}{2}u_2 \\ \hat{\mathrm{S}}_{\mathrm{z}}u_3 &= \frac{1}{2}u_3 \\ \hat{\mathrm{S}}_{\mathrm{z}}u_4 &= -\frac{1}{2}u_4.\end{aligned}$$

4) Helicity

In the Dirac-Pauli representation

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

and since β contains the identity matrix it is clear that $\left[\hat{h}, m\beta\right] = 0$. Therefore, it is only necessary to consider the commutator $\left[\hat{h}, \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\right]$.

$$\begin{aligned}
\left[\hat{h}, \hat{\mathrm{H}}_{\mathrm{D}}\right] = \left[\hat{h}, \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\right] &= \frac{1}{2\mathrm{p}}\left[\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{p}} \end{pmatrix}\right] \\
&= \frac{1}{2\mathrm{p}}\left[\begin{pmatrix} 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 \\ (\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})^2 & 0 \end{pmatrix}\right] \\
&= 0.
\end{aligned}$$