Particle Physics 1 : Exercise 13

1) Hypothetical spin-zero quarks

Since the colour quantum numbers of the quarks have nothing to do with spin, the colour singlet states are still

$$\frac{1}{\sqrt{3}}(r\overline{r} + g\overline{g} + b\overline{b})$$

and

$$\frac{1}{\sqrt{6}}(rgb - grb + gbr - bgr + brg - rgb).$$

Hence, due to colour confinment, we still expect to see *mesons* containing a quark and an antiquark and *baryons* containing three quarks.

Mesons

Since the flavour quantum numbers of the quarks remain unchanged, the flavour wavefunctions for the mesons retain their usual SU(3) form, and we would expect to see the usual flavour nonets. Given that for $s = 0$ there is no spin degree of freedom, there is only a single nonet corresponding to each value of L. The overall parity of a two particle system in a state with orbital angular momentum L is

$$P = P_1 P_2 (-1)^L.$$

Spin-zero quarks would be bosons, and would have the same intrinsic parity, thus $P_1 = P_2$. Hence, for a meson formed from spin-0 quarks :

$$P(q\overline{q}) = (-1)^L.$$

Consequently, one would expect to see nonets with total angular momentum equal to L and parity equal to $(-1)^L$:

$$J^P = 0^+, 1^-, 2^+, 3^-, \ldots,$$

which is in contradiction to the observed meson states.

Baryons

For baryons, which are built up from three “identical” spin-1/2 quarks, with appropriate colour, spin and flavour quantum numbers, the overall wavefunction is

$$\psi = \phi_{\text{flavour}}\, \chi_{\text{spin}}\, \xi_{\text{colour}}\, \eta_{\text{space}}.$$

For spin-1/2 quarks, i.e. fermions, ψ must be totally antisymmetric under interchange of any pair of quarks within the baryon. For baryons made from spin-0 quarks, the wavefunction would become just

$$\psi = \phi_{\text{flavour}}\, \xi_{\text{colour}}\, \eta_{\text{space}}.$$

and since the quarks are now bosons, ψ must be totally symmetric under quark interchange. For spin-0 quarks, the colour wavefunction is the usual SU(3) color singlet, which is totally antisymmetric under the interchange of any pair of quarks within the baryon. Hence $\phi_{\text{flavour}}\, \eta_{\text{space}}$ must now be totally antisymmetric. For $L = 0$ baryons, η_{space} is totally symmetric, so ψ_{flavour} must be totally antisymmetric. The only totally antisymmetric flavour wavefunction which can be constructed out of the three flavours u, d and s is

$$\phi_{\text{flavour}} = \frac{1}{\sqrt{6}}(uds - dus + dsu - sdu + sud - usd).$$

Updated: 20 décembre 2023

Hence, for spin-0 quarks one would expect only a single $L = 0$ baryon state, with the flavour content uds. This baryon would have parity $P = (+1) \cdot (+1) \cdot (+1) \cdot (-1)^0 = +1$ and total spin zero (since $L = 0$ and $S = 0$) giving a

$$J^P = 0^+ \text{ singlet}$$

as the lightest baryon multiplet.

2) Baryon wavefunctions for a colourless model

(a) If the colour did not exist, the baryon wavefunctions would be constructed from

$$\psi = \phi_{\text{flavour}} \chi_{\text{spin}} \eta_{\text{space}}.$$

For the $L = 0$ baryons, the spatial wavefunction is symmetric and the requirement that the overall wavefunction ψ is antisymmetric implies that the combination of $\phi_{\text{flavour}} \chi_{\text{spin}}$ must be antisymmetric under the interchange of any two quarks. In the lecture we have seen that the combination of three up or down quarks (where u and d can be regarded as an SU(2) dublet of isospin 1/2) gives the following SU(2) multiplets

$$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2,$$

corresponding to a quartet with isospin 3/2 and two doublets of isospin 1/2. The flavour wavefunctions of the states in the quartet are totally symmetric under quark interchange, while the two doublets present mixed symmetry : the wavefunctions ϕ_S in one doublet are symmetric under the interchange of the first two quarks $1 \leftrightarrow 2$, while the wavefunctions ϕ_A in the other doublet are antisymmetric under the same interchange $1 \leftrightarrow 2$, but in both doublets there is no defined symmetry under the interchange $2 \leftrightarrow 3$ and $1 \leftrightarrow 3$. The same arguments can be applied to derive the spin states, that are thus divided into a spin-3/2 quartet and two spin-1/2 doublets. As for the isospin, the spin wavefunctions χ_S in one doublet are symmetric under the interchange of the first two quarks, while the wavefunctions χ_A in the other doublet are antisymmetric under the interchange of the same two quarks. The explicit expressions for ϕ_S, ϕ_A, χ_S and χ_A are given in the lecture.

The linear combination

$$\psi = \alpha \phi_S \chi_A + \beta \phi_A \chi_S$$

is antisymmetric under the interchange of quarks $1 \leftrightarrow 2$, and for the right choice of α and β is antisymmetric under the interchange of any two quarks. Using the explicit forms of the flavour and spin wavefunctions for $I_3 = +1/2$ and $S_3 = +1/2$,

$$\begin{aligned}
\psi = &\frac{\alpha}{\sqrt{12}}(2u\uparrow u\downarrow d\uparrow -2u\downarrow u\uparrow d\uparrow -u\uparrow d\downarrow u\uparrow +u\downarrow d\uparrow u\uparrow -d\uparrow u\downarrow u\uparrow +d\downarrow u\uparrow u\uparrow) \\
&+ \frac{\beta}{\sqrt{12}}(2u\uparrow d\uparrow u\downarrow -2d\uparrow u\uparrow u\downarrow -u\uparrow d\downarrow u\uparrow +d\uparrow u\downarrow u\uparrow -u\downarrow d\uparrow u\uparrow +d\downarrow u\uparrow u\uparrow).
\end{aligned}$$

The relation between the coefficients α and β can be found by considering the transformation properties under the interchange of quarks $2 \leftrightarrow 3$ (or $1 \leftrightarrow 3$) :

$$\begin{aligned}
\psi' &= \psi_{2\leftrightarrow 3} \\
&= \frac{\alpha}{\sqrt{12}}(2u\uparrow d\uparrow u\downarrow -2u\downarrow d\uparrow u\uparrow -u\uparrow u\uparrow d\downarrow +u\downarrow u\uparrow d\uparrow -d\uparrow u\uparrow u\downarrow +d\downarrow u\uparrow u\uparrow) \\
&+ \frac{\beta}{\sqrt{12}}(2u\uparrow u\downarrow d\uparrow -2d\uparrow u\downarrow u\uparrow -u\uparrow u\uparrow d\downarrow +d\uparrow u\uparrow u\downarrow -u\downarrow u\uparrow d\uparrow +d\downarrow u\uparrow u\uparrow).
\end{aligned}$$

The requirement of overall antisymmetry implies that $\psi_{2\leftrightarrow 3} = -\psi$. For example, consider the uud parts of the above wavefunctions, the requirement of antisymmetry implies :

$$\begin{aligned} &2\alpha u\uparrow u\downarrow d\uparrow - 2\alpha u\downarrow u\uparrow d\uparrow \\ &= \alpha u\uparrow u\uparrow d\downarrow - \alpha u\downarrow u\uparrow d\uparrow - 2\beta u\uparrow u\downarrow d\uparrow + \beta u\uparrow u\uparrow d\downarrow + \beta u\downarrow u\uparrow d\uparrow, \end{aligned}$$

which is satisfied for $\beta = -\alpha$. Hence

$$\psi = \frac{1}{\sqrt{2}}(\phi_S\chi_A - \phi_A\chi_S),$$

Which with a little straightforward algebra can be re-written as

$$\psi = \frac{1}{\sqrt{6}}(u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + u\downarrow d\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow + d\uparrow u\uparrow u\downarrow).$$

(b) Within the assumed SU(3) flavour symmetry, just as before, the combinations of mixed symmetry flavour and spin states will give rise to a spin-1/2 octet. In addition, it is possible to construct a state where $\phi_{\text{flavour}} \times \chi_{\text{spin}}$ is antisymmetric by combining the totally antisymmetric flavour singlet with a symmetric spin-3/2 state. Hence, in this model there would be an octet of spin-1/2 baryons and a single spin-3/2 uds state.

(c) In this model, the wavefunctions of the "spin-up" proton and neutron are

$$|p\uparrow\rangle = \frac{1}{\sqrt{6}}(u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + u\downarrow d\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow + d\uparrow u\uparrow u\downarrow).$$

and

$$|n\uparrow\rangle = \frac{1}{\sqrt{6}}(d\uparrow d\downarrow u\uparrow - d\downarrow d\uparrow u\uparrow + d\downarrow u\uparrow d\uparrow - u\uparrow d\downarrow d\uparrow - d\uparrow u\uparrow d\downarrow + u\uparrow d\uparrow d\downarrow),$$

where the neutron wavefunction is obtained from the proton's one just by swapping up and down quarks, $u \leftrightarrow d$. Since, for each term, the spins of the two quarks with the same flavour are always opposite, they do not contribute to the overall magnetic moment, which is determined solely by the magnetic moment of the other quark. Hence, taking $m_u \sim m_d$,

$$\frac{\mu_n}{\mu_p} = \frac{\mu_u}{\mu_d} = -2.$$

Therefore, this colourless model does not predict the observed ratio of magnetic moments of the proton and neutron.