Course 08/2

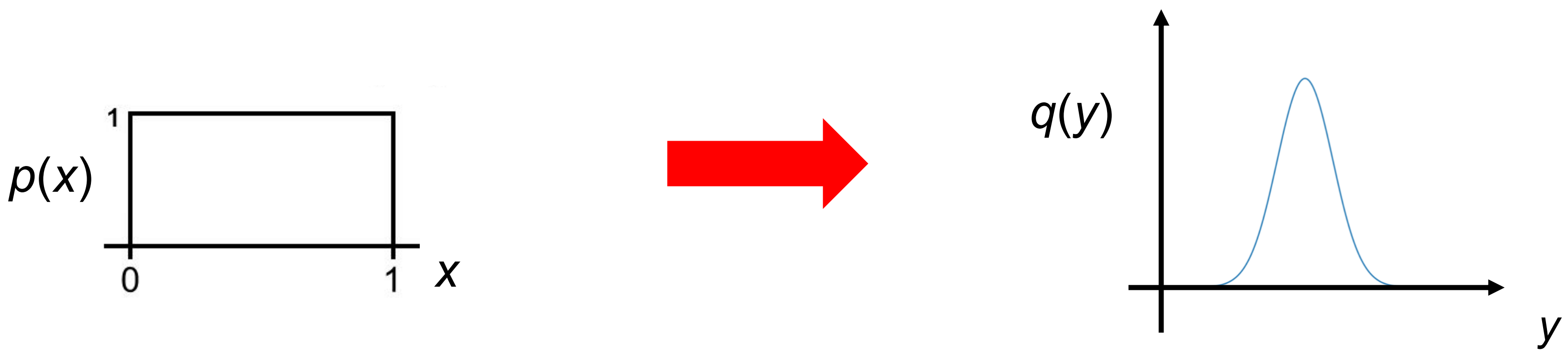
Nonuniform random number distributions

- General formulation
- Example: the exponential distribution
- Example: the Gaussian distribution
    - Application of central limit theorem
    - Method of Box & Muller
- Rejection method of von Neumann

General formulation

Issue

We assume that a random variable x with a uniform distribution is available and we would like to generate a random variable y distributed according to a desired distribution $q(y)$:



Our procedure

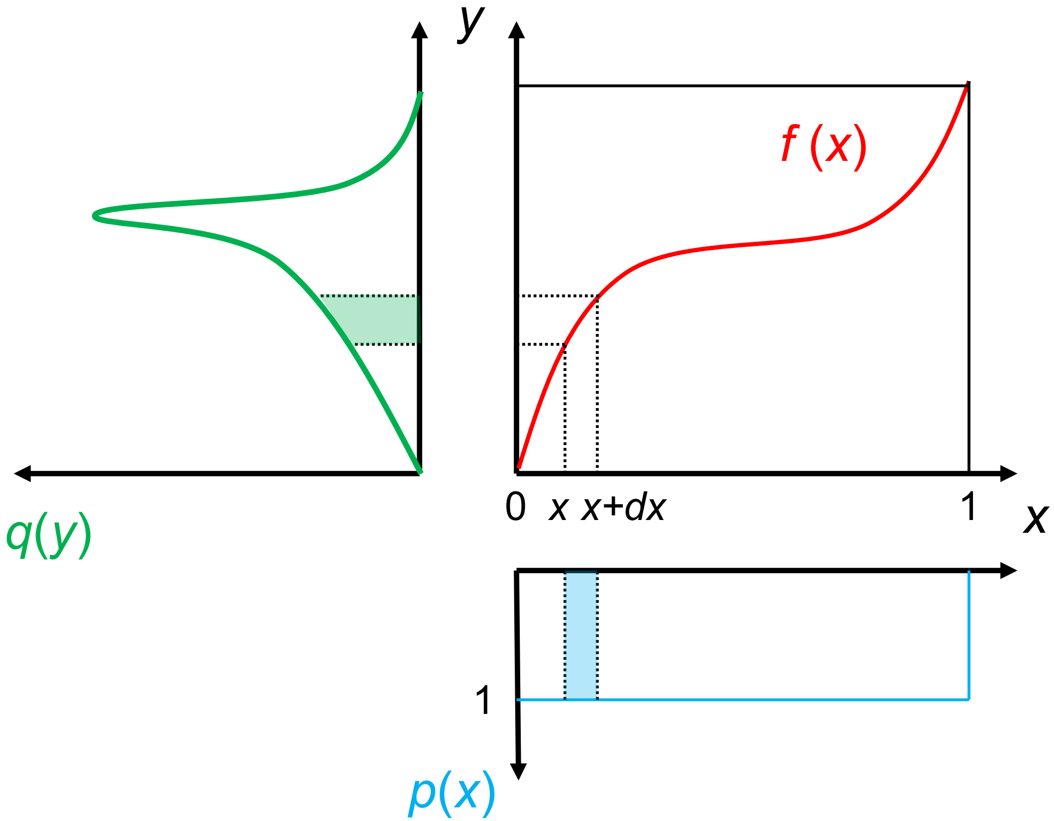
We will address this issue by starting from a simpler problem: Given a new variable y given by

$$y = f(x),$$

what is the distribution $q(y)$ of this variable?

General formulation

Link between $f(x)$ and $q(y)$



Conservation of probability

$$| p(x)\, dx | = | q(y)\, dy |$$

Probability of finding x in $[x, x+dx]$

Probability of finding y in $[y, y+dy]$

$$q(y) = p(x) \left| \frac{dx}{dy} \right|$$

General formulation

$$q(y) \;=\; p(x)\left|\frac{dx}{dy}\right|$$

$y = f(x) \;\rightarrow\; x = f^{-1}(y)$ and $p(x) = 1$

We find

$$q(y) \;=\; \frac{d}{dy}\,[\,f^{-1}(y)\,]$$

In general, $q(y)$ is known since it is the desired distribution and the real question is what is $f(x)$:

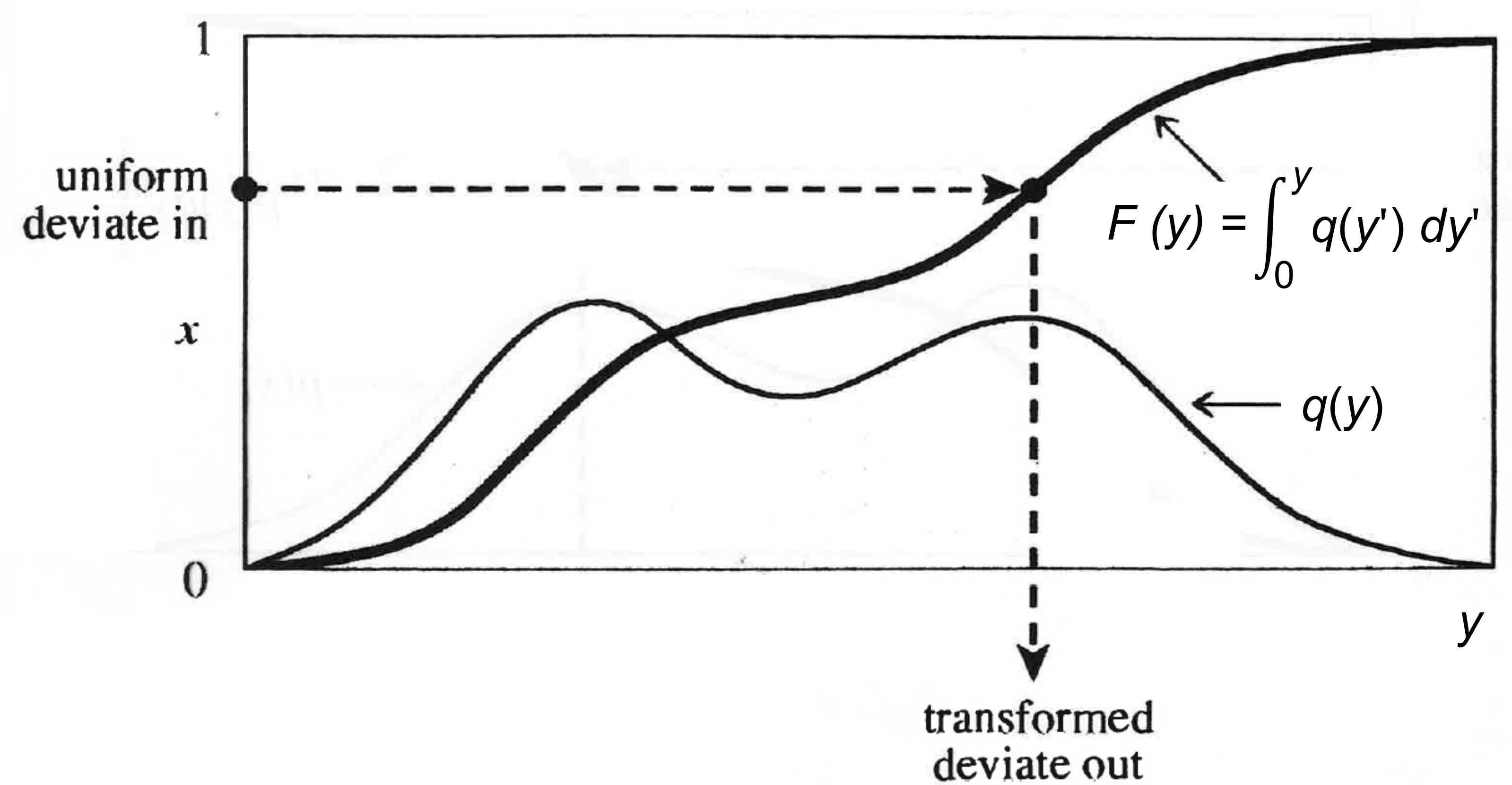
$$\boxed{\int_0^y q(y')\,dy' \;=\; f^{-1}(y) \;=\; x\,(y)}$$

Two steps:

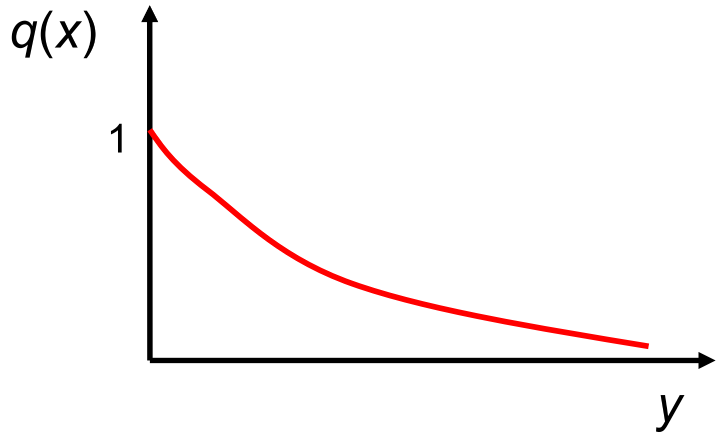
1. Integration of $q(y)$
2. Inversion of the result

General formulation

$$F(y) = \int_0^y q(y')\,dy' = f^{-1}(y) = x(y)$$



Example: the exponential distribution



$$q(y) = e^{-y}$$

What is $f(x)$?

$$\frac{d}{dy}\left[f^{-1}(y) \right] = e^{-y}$$

$$f^{-1}(y) = -e^{-y} + \text{const.}$$

$$x = -e^{-f(x)} + \text{const.}$$

$$f(x) = -\ln(\text{const.} - x)$$

$$\boxed{f(x) = -\ln(1 - x)}$$

The constant can be found by imposing:

$$x \to y \;:\; (0, 1) \to (0, \infty)$$

Example: the exponential distribution

$$f(x) = -\ln(1 - x)$$

Another way of expressing this finding starts directly from the "conservation of probability":

$$|\, p(x)\, dx \,| = |\, q(y)\, dy \,|$$

$p(x) = 1$	$q(y) = e^{-y}$
uniform	exponential

This corresponds to the change of variable: $y = -\ln(1 - x)$

Example: the Gaussian distribution

$$q(y) = \frac{1}{\sqrt{2\pi}} \, e^{-y^2/2} \qquad (\mu = 0, \sigma = 1)$$

Application of central limit theorem

An approximate Gaussian variable can be obtained by making use of the central limit theorem, but many variables are necessary for obtaining a single Gaussian one.

Application of the general formulation

The general formulation gives:

$$\frac{d}{dy}\left[f^{-1}(y) \right] = \frac{1}{\sqrt{2\pi}} \, e^{-y^2/2}$$

This requires first the integration of the Gaussian function, which gives an error function. Second, the error function needs to be inverted. This is an impractical procedure, which cannot be carried out analytically.

Example: the Gaussian distribution

Method of Box and Muller (1958)

The idea is finding a distribution in two dimensions, which is Gaussian in both variables:

$$Q\,(y_1, y_2) = q\,(y_1)\;q\,(y_2)$$

where $$q(y) = \frac{1}{\sqrt{2\pi}}\,e^{-y^2/2}$$ (normalized Gaussian distribution)

Procedure

We are going to define a series of transformations of variables until y_1 and y_2 are expressed in terms of uniform variables.

Example: the Gaussian distribution

$$Q(y_1, y_2)\, dy_1\, dy_2 = \frac{1}{2\pi}\, e^{-y_1^2/2}\, e^{-y_2^2/2}\, dy_1\, dy_2$$

Transformation to polar variables: $y_1 = r\cos\phi$ and $y_2 = r\sin\phi$

$$Q(y_1, y_2)\, dy_1\, dy_2 = e^{-r^2/2}\, r\, dr\ \frac{1}{2\pi}\, d\phi$$

$$= p(r)\, \mathrm{d}r \quad p(\phi)\, d\phi$$

Transformation to $u = r^2/2$

$$Q(y_1, y_2)\, dy_1\, dy_2 = e^{-u}\, du\ \frac{1}{2\pi}\, d\phi$$

$$= \underbrace{p(u)\, \mathrm{d}u}_{= e^{-u}} \quad \underbrace{p(\phi)\, d\phi}_{= 1/(2\pi)}$$

Example: the Gaussian distribution

$$Q\,(\,y_1,\, y_2\,)\; dy_1\, dy_2 \quad = \quad e^{-u} \quad du \;\; \frac{1}{2\pi}\, d\phi$$

In these new variables: $y_1 = \sqrt{2u}\cos\phi$ and $y_2 = \sqrt{2u}\sin\phi$

We now transform the variable u to the uniform variable x, as seen in the example for the exponential distribution:

$$u \;=\; -\ln\,(\,1 - \;x\,)$$

$$Q\,(\,y_1,\, y_2\,)\; dy_1\, dy_2 \quad = \quad \underbrace{p(x)}_{=\,1}\, dx \quad \underbrace{p(\phi)}_{=\,1/(2\pi)}\, d\phi$$

Global transformation:

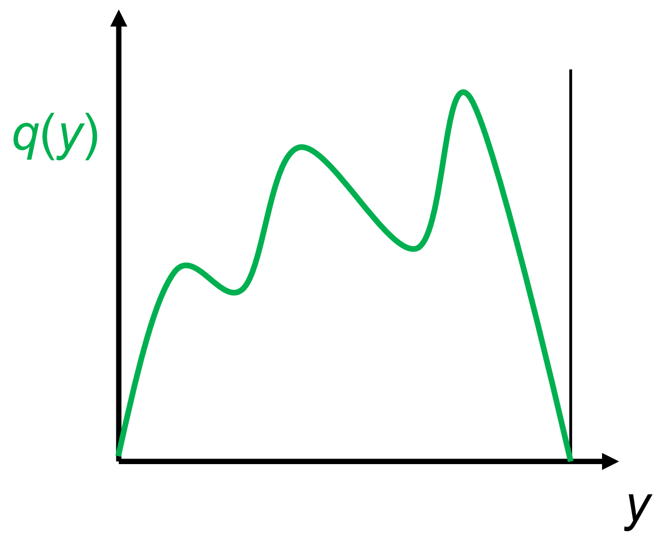
$$\boxed{\begin{aligned} y_1 &= \sqrt{-2\,\ln(1-x)}\,\cos\phi \\ y_2 &= \sqrt{-2\,\ln(1-x)}\;\;\sin\phi \end{aligned}}$$

Rejection method of von Neumann (1951)

Target Generate random variables according to a probability density $q(y)$:

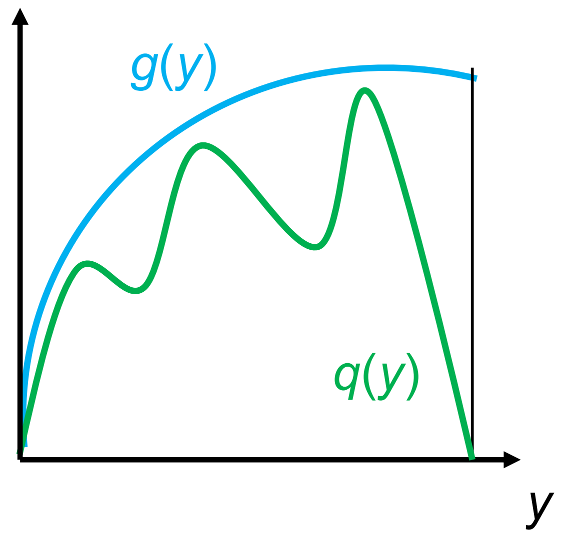
$$q(y) \geq 0 \qquad \int_{-\infty}^{\infty} q(y)\, dy = 1$$

It is assumed that proceeding through the inverse of the integral of $q(y)$ is impractical.



Rejection method of von Neumann (1951)

Procedure



1. Find a *majorizing* function $g(y)$, such that $g(y) \geq q(y)$ for all y of interest.

   The associated probability density is given by

$$g^N(y) = \frac{g(y)}{\int_{-\infty}^{\infty} g(y)\, dy}$$

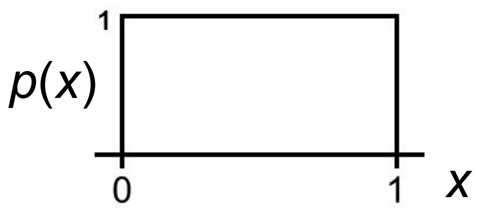
   The function $g(y)$ is chosen in such a way that random variables distributed according to $g^N(y)$ are easily available.

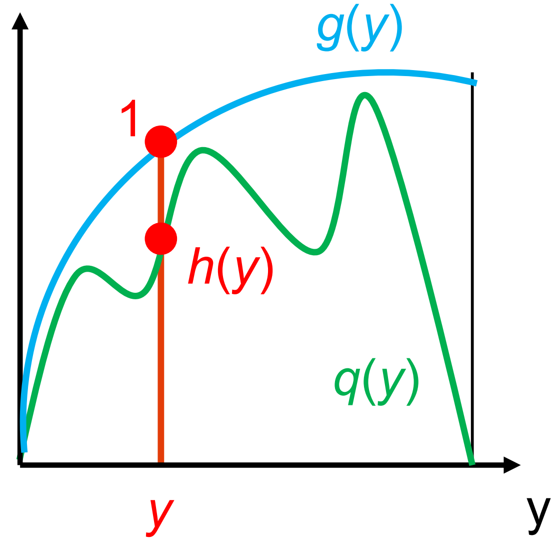
Rejection method of von Neumann (1951)

Procedure

2. Generate a random variate y from $g^N(y)$.

3. Generate an auxiliary random variate x from the uniform distribution $p(x)$:





4. Define $h(y) = q(y) / g(y)$ (fraction ≤ 1)

5. if $x \leq h(y)$ $\Rightarrow$ take the random variate $z = y$; back to step 2.

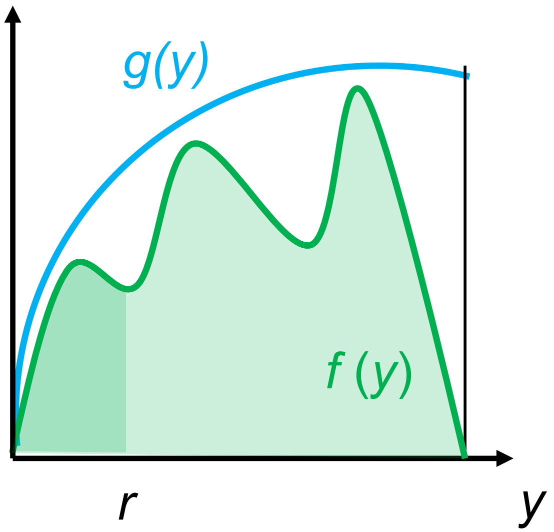
   if $x > h(y)$ $\Rightarrow$ back to step 2.

$\Rightarrow$ the random variate z is distributed according to $q(z)$.

Rejection method of von Neumann (1951)

Proof

Let us focus on the probability $P\,(\,z \leq r\,) \;=\; P\,(\,y \leq r \;|\; x \leq h(y)\,)$



$$= \frac{\text{Accepted } y \leq r \text{ (dark shaded)}}{\text{All accepted } y \text{ (dark + light shaded)}}$$

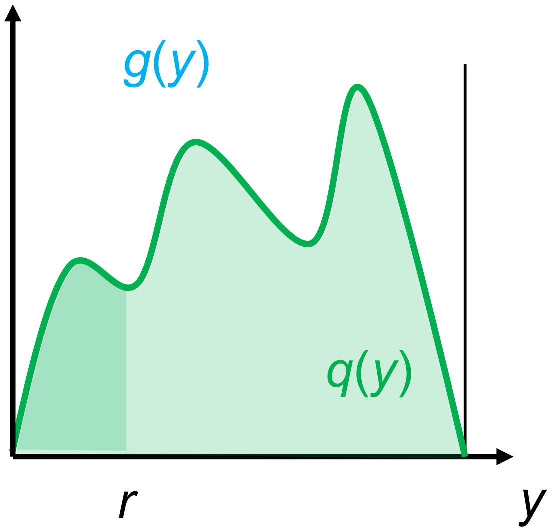
$$= \frac{\int_{-\infty}^{r} dy \;\; g^N(y) \int_0^{h(y)} p(x)\, dx}{\int_{-\infty}^{\infty} dy \;\; g^N(y) \int_0^{h(y)} p(x)\, dx}$$

Rejection method of von Neumann (1951)

$$P(z \leq r) = \frac{\int_{-\infty}^{r} dy \;\; g^N(y) \int_0^{h(y)} p(x)\, dx}{\int_{-\infty}^{\infty} dy \;\; g^N(y) \int_0^{h(y)} p(x)\, dx} \underset{p(x)\,=\,1}{=} \frac{\int_{-\infty}^{r} dy \;\; g^N(y)\, h(y)}{\int_{-\infty}^{\infty} dy \;\; g^N(y)\, h(y)}$$

$$\boxed{g^N(y) = \frac{g(y)}{\int_{-\infty}^{\infty} g(y)\, dy}}$$

$$\boxed{h\,(y) = \frac{q(y)}{g(y)}}$$



$$= \frac{\int_{-\infty}^{r} dy \; \frac{\cancel{g(y)}}{\cancel{\int_{-\infty}^{\infty} g(y)\, dy}} \; \frac{q(y)}{\cancel{g(y)}}}{\int_{-\infty}^{\infty} dy \; \frac{\cancel{g(y)}}{\cancel{\int_{-\infty}^{\infty} g(y)\, dx}} \; \frac{q(y)}{\cancel{g(y)}}} = \int_{-\infty}^{r} dy \;\; q(y)$$

$$\boxed{\int_{-\infty}^{\infty} q(y)\, dy = 1}$$

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