Course 01/2

Ordinary differential equations (1/2)

- General
- Euler method
- Taylor method
- Multistep method
- Implicit methods

General

Most general form

$$\frac{d\vec{y}}{dx} = f(x, \vec{y})$$

If higher order, e.g. $m\frac{d^2z}{dt^2} = F(z)$

we define the momentum $p(t) = m\frac{dz}{dt}$

and we obtain two coupled first-order equations: $\frac{dz}{dt} = \frac{p}{m}$; $\frac{dp}{dt} = F(z)$

Since the matrix-structure is natural, we only consider the following as the most general form:

$$\boxed{\frac{dy}{dx} = f(x, y)}$$

Problem

$$\frac{dy}{dx} = f(x, y)$$

Initial value problem: $y(x = 0) = y_0$ What is the evolution of y ?

General strategy

N steps

1. Discretization

$x_n = n\,h$ $h = 1 / N$

2. Recursion formula: y_n as a function of y_{n-1}, y_{n-2} …

Euler method

$$\frac{dy}{dx} \quad \rightarrow \quad \frac{y_{n+1} - y_n}{h} \quad + \quad o\,(\,h\,)$$

$$\frac{dy}{dx} = f\,(\,x\,,\,y\,) \quad \rightarrow \quad y_{n+1} \;=\; y_n \;+\; h\,f\,(\,x_n\,,\,y_n\,) \;+\; o\,(\,h^2\,)$$

after N steps $\rightarrow$ $o\,(\,h\,)$

Example: $\dfrac{dy}{dx} = -\,x\,y \quad ; \quad y\,(\,0\,) = 1$

Analytical solution: $y \;=\; \mathrm{e}^{-\frac{1}{2}x^2}$

Euler method: $y_{n+1} \;=\; y_n \;-\; h\,x_n\,y_n + \; o\,(\,h^2\,)$

Results

$y(1) = 0.606531$ $y(3) = 0.011109$

Exact solution : $y = e^{-\frac{1}{2}x^2}$

Table 2.1 Error in integrating $dy/dx = -xy$ with $y(0)=1$

h	Euler's method Eq. (2.6) $y(1)$	$y(3)$
0.500	-.143469	.011109
0.200	-.046330	.006519
0.100	-.021625	.003318
0.050	-.010453	.001665
0.020	-.004098	.000666
0.010	-.002035	.000333
0.005	-.001014	.000167
0.002	-.000405	.000067
0.001	-.000203	.000033

Fractional error: 0.0033 0.030

Euler method

$$y_{n+1} = y_n + h\,f(x_n, y_n) + o(h^2)$$

- Considered to be of low accuracy.
- With a higher accuracy larger steps would be allowed.

Taylor method

$$y_{n+1} = y(x_n + h) = y_n + h\, y_n' + \tfrac{1}{2} h^2 y_n'' + o(h^3)$$

$$y_n' = f(x_n, y_n)$$ differential equation

$$y_n'' = \frac{df}{dx}(x_n, y_n) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

Upon replacing in the recursion relation, we get:

$$y_{n+1} = y_n + h f + \tfrac{1}{2} h^2 \left[\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right] + o(h^3)$$

where all the derivatives are evaluated at (x_n, y_n)

Limitation: the function must be simple to differentiate

Advantage: higher accuracy than with Euler method

Results

$y(1) = 0.606531$ $y(3) = 0.011109$

Exact solution : $y = e^{-\frac{1}{2}x^2}$

Table 2.1 Error in integrating $dy/dx = -xy$ with $y(0)=1$

	Euler's method Eq. (2.6)		Taylor series Eq. (2.10)	
h	$y(1)$	$y(3)$	$y(1)$	$y(3)$
0.500	-.143469	.011109	.032312	-.006660
0.200	-.046330	.006519	.005126	-.000712
0.100	-.021625	.003318	.001273	-.000149
0.050	-.010453	.001665	.000317	-.000034
0.020	-.004098	.000666	.000051	-.000005
0.010	-.002035	.000333	.000013	-.000001
0.005	-.001014	.000167	.000003	.000000
0.002	-.000405	.000067	.000001	.000000
0.001	-.000203	.000033	.000000	.000000

Fractional error: 0.0033 0.030

Multistep methods

$$\frac{dy}{dx} = f(x, y) \qquad\qquad y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

Problem: f unknown over integration interval

Possible solution: linear extrapolation

$$f \cong \frac{(x - x_{n-1})}{h} f_n - \frac{(x - x_n)}{h} f_{n-1} + o(h^2)$$

The integration gives

$$y_{n+1} = y_n + h\left(\frac{3}{2} f_n - \frac{1}{2} f_{n-1}\right) + o(h^3)$$

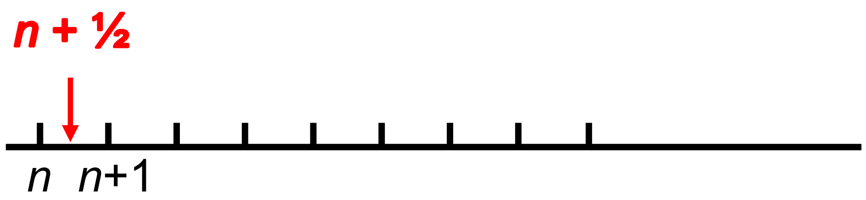
Adams-Bashforth 2-point formula

Possible extension: the extrapolation can be performed with higher order polynomials using more points.

Disadvantage: multistep….the start has to occur with some other method, e.g. with the Taylor method or the Runga-Kutta method

Implicit methods

So far y_n has been explicitly given through the available information.
In implicit methods, an equation has to be solved to find y_n .



$$x_{n+½} = (n + ½)\, h$$

$$\left.\frac{dy}{dx}\right|_{x_{n+½}} = f\ (\, x_{n+½}\, ,\, y_{n+½}\,)$$

Implicit methods

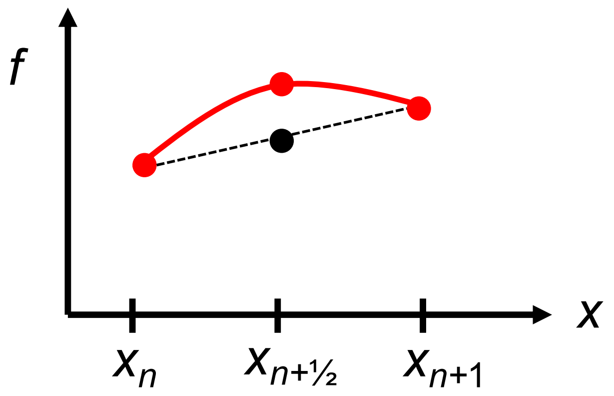
Left-hand side: symmetric difference

$$y_{n+1} = y\,(x_{n+½} + ½\,h) = y\,(x_{n+½}) + ½\,h\,y'(x_{n+½}) + ½\,(½\,h)^2\,y''(x_{n+½}) + o\,(\,h^3\,)$$

$$y_n = y\,(x_{n+½} - ½\,h) = y\,(x_{n+½}) - ½\,h\,y'(x_{n+½}) + ½\,(½\,h)^2\,y''(x_{n+½}) + o\,(\,h^3\,)$$

$$\Rightarrow \quad y'(x_{n+½}) = \left.\frac{dy}{dx}\right|_{x_{n+½}} = \frac{y_{n+1} - y_n}{h} + o\,(\,h^2\,)$$

Right-hand side: linear interpolation



$$\Rightarrow \quad f\,(\,x_{n+½}\,,\,y_{n+½}\,) = ½\,[\,f\,(\,x_n\,,\,y_n\,) + f\,(\,x_{n+1}\,,\,y_{n+1}\,)\,] \quad + o\,(\,h^2\,)$$

Implicit methods

we obtain

$$y_{n+1} = y_n + \tfrac{1}{2} h \left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right] + o(h^3)$$

implicit equation for y_{n+1}

In particular, when f is linear in y : $f(x, y) = g(x) \cdot y$

$$y_{n+1} = \left[\frac{1 + \tfrac{1}{2} g(x_n) h}{1 - \tfrac{1}{2} g(x_{n+1}) h} \right] y_n$$

In our example: $\dfrac{dy}{dx} = -xy$ i.e. $g(x) = -x$

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Exact solution : $y = e^{-\frac{1}{2}x^2}$

Table 2.1 Error in integrating $dy/dx = -xy$ with $y(0) = 1$

	Euler's method Eq. (2.6)		Taylor series Eq. (2.10)		Implicit method Eq. (2.18)	
h	$y(1)$	$y(3)$	$y(1)$	$y(3)$	$y(1)$	$y(3)$
0.500	-.143469	.011109	.032312	-.006660	-.015691	.001785
0.200	-.046330	.006519	.005126	-.000712	-.002525	.000255
0.100	-.021625	.003318	.001273	-.000149	-.000631	.000063
0.050	-.010453	.001665	.000317	-.000034	-.000157	.000016
0.020	-.004098	.000666	.000051	-.000005	-.000025	.000003
0.010	-.002035	.000333	.000013	-.000001	-.000006	.000001
0.005	-.001014	.000167	.000003	.000000	-.000001	.000000
0.002	-.000405	.000067	.000001	.000000	.000000	.000000
0.001	-.000203	.000033	.000000	.000000	.000000	.000000

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