

Classical Electrodynamics

Week 9

1. Consider an infinite cylinder of radius a carrying the current I . The cylinder is surrounded by an insulator with magnetic permeability μ . A metallic cylindrical surface of radius $b > a$ conducts the current in the opposite direction.

Determine the magnetic field $\mathbf{H}$, the magnetic induction $\mathbf{B}$ and the magnetization $\mathbf{M}$ in every point in space. Find the free current density $\mathbf{J}$ and the average microscopic current density $\langle \mathbf{j} \rangle$.

Solution

The Maxwell equations of magnetostatics are

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J}\end{aligned}\tag{1}$$

For $a < r < b$, when we calculate the circulation of H along a circle of radius r we obtain

$$\int \mathbf{H} \cdot d\mathbf{l} = \int \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int \mathbf{J} \cdot d\mathbf{S} = I ,\tag{2}$$

where we have used the Stokes theorem, the second equation above and the definition of current density. Since we know, because of the cylindrical symmetry, that $\mathbf{H} = H(r)\mathbf{e}_\phi$, we can write

$$\int \mathbf{H} \cdot d\mathbf{l} = H(r)2\pi r ,\tag{3}$$

and thus

$$\mathbf{H} = \frac{I}{2\pi r} \mathbf{e}_\phi .\tag{4}$$

For $r > b$ the total current is zero, so we have $\mathbf{H} = 0$. For $r < a$, the current relevant for H becomes $J\pi r^2 = \frac{Ir^2}{a^2}$. Now that we have $\mathbf{H}$, we can calculate $\mathbf{B}$ using

$$\mathbf{B} = \mu \mathbf{H} .\tag{5}$$

The magnetic induction is

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I r}{2\pi a^2} \mathbf{e}_\phi & \text{for } r < a \\ \mathbf{B} &= \frac{\mu I}{2\pi r} \mathbf{e}_\phi & \text{for } a < r < b \\ \mathbf{B} &= 0 & \text{for } r > b ,\end{aligned}\tag{6}$$

and the magnetization is given by $\mathbf{M} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{H}$ and we obtain as expected a non-vanishing component only in the insulator

$$\begin{aligned}\mathbf{M} &= 0 & \text{for } r < a \\ \mathbf{M} &= \frac{I}{2\pi r} \frac{\mu - \mu_0}{\mu_0} \mathbf{e}_\phi & \text{for } a < r < b \\ \mathbf{M} &= 0 & \text{for } r > b .\end{aligned}\tag{7}$$

In order to find the average microscopic current density we start by writing $\mathbf{M}$ in a more synthetic way:

$$\frac{I}{2\pi r} \frac{\mu - \mu_0}{\mu_0} \mathbf{e}_\phi (\Theta(r - a) - \Theta(r - b)) . \quad (8)$$

We can now apply the formula

$$\langle \mathbf{j} \rangle = \mathbf{J} + \nabla \times \mathbf{M} , \quad (9)$$

and when we do so, we must remember that the derivative of the Heaviside function Θ is the Dirac δ , as we have shown in a previous exercise. We obtain

$$\nabla \times \mathbf{M} = \frac{1}{r} \frac{\partial}{\partial r} (rM(r)) \mathbf{e}_z = \frac{I}{2\pi r} \frac{\mu - \mu_0}{\mu_0} \mathbf{e}_z (\delta(r - a) - \delta(r - b)) . \quad (10)$$

Note that, since $\mu > \mu_0$ for a paramagnetic insulator and $\mu < \mu_0$ for a diamagnetic insulator, we can see that a paramagnetic insulator gives a current in the same direction of $\mathbf{J}$, so that $\mathbf{B}$ is increased, whereas in a diamagnetic insulator the direction is the opposite and $\mathbf{B}$ is decreased.

2. In electrostatics, the n -th pole is given by

$$Q_{i_1 \dots i_n} = \int d^3x \rho(\mathbf{x}) T_{i_1 \dots i_n}(\mathbf{x}) , \quad (11)$$

where the totally symmetric tensor $T_{i_1 \dots i_n}$ can be defined by

$$T_{i_1 \dots i_n} = (2n - 1)!! x_{i_1} \dots x_{i_n} - A_{i_1 \dots i_n} , \quad (12)$$

with the double factorial $(2n - 1)!! = (2n - 1)(2n - 3)(2n - 5) \dots (5)(3)(1)$ and $(-1)!! = 1$. The tensor $A_{i_1 \dots i_n}$ is an homogeneous polynomial of degree n in the components of $\mathbf{x}$ and it contains at least one Kronecker- δ so that the trace vanishes:

$$T_{i_1 \dots i_n} \delta_{i_k i_l} = 0 , \quad \forall k, l \in \{1, 2, \dots, n\} , \quad k \neq l . \quad (13)$$

It is convenient to introduce the notation $B_{(i_1 \dots i_n)}$ for the total symmetrization of a tensor $B_{i_1 \dots i_n}$. More precisely,

$$B_{(i_1 \dots i_n)} \equiv \frac{1}{n!} \sum_{\text{perm } \sigma} B_{\sigma(i_1 \dots i_n)} , \quad (14)$$

where the sum runs over all permutations σ of the n indices $i_1 \dots i_n$. For example,

$$B_{(ij)} \equiv \frac{1}{2} (B_{ij} + B_{ji}) , \quad v_{(i} w_{j)} \equiv \frac{1}{2} (v_i w_j + v_j w_i) \quad (15)$$

$$B_{(ijk)} \equiv \frac{1}{6} (B_{ijk} + B_{ikj} + B_{jik} + B_{jki} + B_{kij} + B_{kji}) . \quad (16)$$

- a) Argue that the tensors $A_{i_1 \dots i_n}$, for $n = 2, 3, 4$, must be of the form

$$A_{ij} = c_2 x^2 \delta_{ij} \quad (17)$$

$$A_{ijk} = c_3 x^2 \delta_{(ij} x_{k)} \quad (18)$$

$$A_{ijkl} = c_4 x^2 \delta_{(ij} x_k x_{l)} + c'_4 x^4 \delta_{(ij} \delta_{kl)}, \quad (19)$$

where c_2, c_3, c_4 and c'_4 are numerical constants.

- b) Determine the constants c_2, c_3, c_4 and c'_4 imposing the trace condition (13).

- c) How many independent components does the tensor $Q_{i_1 \dots i_n}$ have? Start by working out the cases $n = 0, 1, 2$. Can you guess the formula for general n ?

Hint: Start by counting the number of independent components in a totally symmetric tensor with n indices and then impose the trace constraint.

- d) Generalize the previous question to d space dimensions. Show that the number of independent components of a traceless symmetric tensor $Q_{i_1 \dots i_n}$ in d dimensions (*i.e.*, each index can take the values $1, 2, \dots, d$) is given by

$$(2n + d - 2) \frac{(n + d - 3)!}{n!(d - 2)!}. \quad (20)$$

Solution

- a) The building blocks for the tensors $A_{i_1 \dots i_n}$ are two elementary tensors: δ_{ij} and x_j . We need to put at least one δ , and we need to build a tensor with n indices. The most general structure is obtained by multiplying k Kronecker- δ s and $n - 2k$ components of $\mathbf{x}$, with $k = 1, 2, \dots, [n/2]$, $[n/2]$ being the integer part of $n/2$. This yields tensors with n indices, and degree $n - 2k$ in the components of $\mathbf{x}$. To compensate, we multiply the result by x^{2k} . Finally, we symmetrize the indices as described in the text. This fixes the rank n tensor up to $[n/2]$ coefficients $c_{k,n}$:

$$A_{i_1 \dots i_n} = \sum_{k=1}^{[n/2]} c_{k,n} x^{2k} \underbrace{\delta_{i_1 i_2} \dots \delta_{i_{2k-1} i_{2k}}}_k \underbrace{x_{i_{2k+1}} \dots x_{i_n}}_{n-2k}. \quad (21)$$

Choosing $n = 2, 3, 4$ we get the specific cases shown in the text. You may still object: why did we write x^{2k} rather than picking some specific components, say $(x_1)^{2k}$? The resulting tensor would have been of the right degree in $\mathbf{x}$. The easy answer is that the purpose in life of $A_{i_1 \dots i_n}$ is to cancel the traces from the tensor $x_{i_1} \dots x_{i_n}$. It is easy to see that all traces of the latter generate factors of x^2 and not specific components. There is also a fancier explanation. The tensor $T_{i_1 \dots i_n}$ in eq. (2) appears in the Taylor expansion of a function of $\mathbf{x}$ and $\mathbf{x}'$:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} T^{i_1 \dots i_n}(\mathbf{x}') \frac{x_{i_1} \dots x_{i_n}}{x^{2n+1}}. \quad (22)$$

The function on the left hand side is invariant under rotations of the reference frame, so each term on the right hand side must be as well. Specific components of the vector $\mathbf{x}'$ are not invariant under rotations, of course, and so cannot appear. Only scalar products can appear on the right hand side. In particular, $x'^2 = \mathbf{x}' \cdot \mathbf{x}'$ is allowed, as are $\mathbf{x} \cdot \mathbf{x}'$ and x^2 .

b) Let's start with $n = 2$:

$$(2n - 1)!! \delta^{ij} x_i x_j - c_2 x^2 \delta_{ij} = 3x^2 - 3c_2 x^2 = 0, \quad (23)$$

that is,

$$c_2 = 1. \quad (24)$$

Starting from $n = 3$, we need to symmetrize. Since δ_{ij} is already symmetric, we do not need to add up all nine permutations of the three indices:

$$\delta_{(ij)x_k} = \frac{1}{3} (\delta_{ij} x_k + \delta_{ik} x_j + \delta_{jk} x_i). \quad (25)$$

Therefore,

$$5 \cdot 3 x^2 x_k = \frac{c_3}{3} x^2 (3x_k + x_k + x_k), \quad (26)$$

that is,

$$c_3 = 9. \quad (27)$$

In the $n = 4$ case, the permutations of four elements are 24, but we can again use symmetry to reduce the needed number. $\delta_{ij} x_k x_l$ is symmetric under the exchanges $i \leftrightarrow j$ and $k \leftrightarrow l$, so we only need 6 permutations:

$$\delta_{(ij)x_k x_l} = \frac{1}{6} (\delta_{ij} x_k x_l + \delta_{ik} x_j x_l + \delta_{il} x_k x_j + \delta_{jk} x_i x_l + \delta_{jl} x_k x_i + \delta_{kl} x_i x_j). \quad (28)$$

Finally, $\delta_{ij} \delta_{kl}$ has the same symmetries of the previous case, plus the exchange $(ij) \leftrightarrow (kl)$. We only need 3 permutations:

$$\delta_{(ij)\delta_{kl}} = \frac{1}{3} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (29)$$

Taking the trace of T_{ijkl} leads to the following:

$$7 \cdot 5 \cdot 3 x_k x_l = \frac{c_4}{6} (7x_k x_l + \delta_{kl} x^2) + \frac{5}{3} c'_4 x^2 \delta_{kl}. \quad (30)$$

This implies

$$c_4 = 90, \quad c'_4 = -\frac{c_4}{10} = -9. \quad (31)$$

c) and d) We directly give the solution for general dimension. Since the tensor $T_{i_1 \dots i_n}$ is symmetric, we are free to order the indices, say in increasing order, and count all the components of the kind

$$T_{\underbrace{1 \dots 1}_{k_1 \text{ times}} \underbrace{2 \dots 2}_{k_2 \text{ times}} \dots \underbrace{d \dots d}_{k_d \text{ times}}}, \quad \sum_{i=1}^d k_i = n. \quad (32)$$

In other words, we need to count the partitions of n in a maximum of d addends - the k'_i s are allowed to be zero. The problem can be rephrased in a way which makes the solution easy. Let us draw a wall each time we increase the value of the index:

$$\underbrace{\boxed{1} \dots \boxed{1} \boxed{1} \boxed{2} \dots \boxed{2} \boxed{1} \dots \boxed{1} \boxed{d} \dots \boxed{d}}_{n+d-1 \text{ boxes}} \quad (33)$$

We include the $d-1$ walls in the count of the boxes. Now, different partitions are completely specified by the position of the walls. So here is the new problem: count in how many ways we can choose $d-1$ boxes out of $n+d-1$. But this is just the definition of a binomial, so we arrive at the following partial answer:

$$\begin{array}{l} \# \text{ of independent components of} \\ \text{a symmetric tensor of rank } n \text{ in } d \\ \text{dimensions} \end{array} = \binom{n+d-1}{d-1} = \frac{(n+d-1)!}{n!(d-1)!}.$$

In setting the traces to zero, we can again take advantage of the symmetry. Once the following equation is satisfied:

$$\delta^{i_1 i_2} T_{i_1 i_2 i_3 \dots i_n} = 0, \quad (34)$$

all other traces are zero as well. Formula (34) amounts to a set of homogeneous linear equations for the components of the tensor. The number of equations is equal to the number of independent ways to choose the indices i_3 to i_n which are not contracted. This is the number of independent components of a symmetric tensor of rank $n-2$. Each equation provides a constraint, and so we get the final answer by subtracting the number of constraints:

$$\begin{array}{l} \# \text{ of independent components of} \\ \text{a symmetric traceless tensor of} \\ \text{rank } n \text{ in } d \text{ dimensions} \end{array} = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} \\ = (2n+d-2) \frac{(n+d-3)!}{n!(d-2)!}.$$

3. A dielectric sphere of radius a and permittivity ε_1 is placed in a constant electric field $\mathbf{E}_0$ in vacuum.

a) Assume that the electrostatic potential can be written as

$$\Phi = f(r) \cos \theta, \quad (35)$$

using spherical coordinates centred at the sphere and with the north pole direction $\theta = 0$ defined by the background electric field $\mathbf{E}_0$.

- i. What is $f(r)$ in the absence of the sphere?
- ii. What are the interface matching conditions for the electromagnetic fields $\mathbf{E}$ and $\mathbf{D}$ in the absence of free charges? What conditions do these imply for $f(r)$ at $r = a$?
- iii. Determine the function $f(r)$ for all $r > 0$.

Hint: Firstly, list the conditions the function $f(r)$ satisfies at $r = 0$, $r = a$ and $r = \infty$. Secondly, derive differential equations for $f(r)$ in the regions $r < a$ and $r > a$. Thirdly, notice that $r \frac{d}{dr} r^l \propto r^l$.

b) Study the potential at large distances to show that the electric dipole of the sphere is given by

$$\mathbf{d} = \alpha \mathbf{E}_0, \quad (36)$$

and determine α .

- c) Consider now a dilute gas of these small dielectric spheres. Let n be the number of spheres per unit volume. Determine the effective electric permittivity ε of this gas relevant to describe its electromagnetic properties at macroscopic length scales $r \gg n^{-\frac{1}{3}} \gg a$.

Solution

- a) i. When the sphere is not there, the electric field is $\mathbf{E} = \mathbf{E}_0 = E_0 \mathbf{e}_z$. The potential is

$$\Phi = -E_0 z = -E_0 r \cos \theta,$$

so $f(r) = -E_0 r$.

- ii. The matching conditions at the interface are:

$$(\mathbf{D}_1 - \mathbf{D}_2)_\perp = \mathbf{0}, \quad (37)$$

$$(\mathbf{E}_1 - \mathbf{E}_2)_\parallel = \mathbf{0}, \quad (38)$$

where the fields $\mathbf{E}_1, \mathbf{D}_1$ are outside the dielectric sphere and the fields $\mathbf{E}_2, \mathbf{D}_2$ are inside. Note that the r.h.s. of the first condition is generally $\rho_s \mathbf{n}$, where $\mathbf{n}$ is a unit normal vector pointing outwards the sphere, and ρ_s is the surface density of free charges, which is 0 in our case. We relate the fields $\mathbf{E}$ and $\mathbf{D}$ with

$$\mathbf{D}_1 = \varepsilon_0 \mathbf{E}_1, \quad \mathbf{D}_2 = \varepsilon_1 \mathbf{E}_2. \quad (39)$$

To get a condition on the potential, we use:

$$\mathbf{E} = -\nabla \Phi = -f'(r) \cos \theta \mathbf{e}_r + \frac{f(r)}{r} \sin \theta \mathbf{e}_\theta.$$

This reduces (37) and (38) to

$$\varepsilon_1 f'(a^-) - \varepsilon_0 f'(a^+) = 0, \quad (40)$$

$$f(a^+) - f(a^-) = 0, \quad (41)$$

where $f(a^\pm)$ means $\lim_{r \rightarrow a^\pm} f(r)$.

- iii. We now solve for the potential everywhere. The boundary conditions at $r = a$ are given by (40),(41). At $r = 0$, the potential should actually not depend on θ so we impose

$$f(0) = 0. \quad (42)$$

When we are really far away of the sphere, its effect should vanish and the potential should converge to the result of question a)

$$\lim_{r \rightarrow \infty} (f(r) - (-E_0 r)) = 0. \quad (43)$$

We use the Maxwell equation:

$$\nabla \cdot \mathbf{D} = \rho(x) = 0$$

which, using (39), indicates we simply have to solve the Laplace equation $\nabla^2 \Phi = 0$, separately in regions $r < a$ and $r > a$.

$$\nabla^2 \Phi = \left(f''(r) + \frac{2}{r} f'(r) - \frac{2}{r^2} f(r) \right) \cos \theta = 0.$$

This must be verified for any θ . Making the Ansatz $f(r) \propto r^l$, we get that the parenthesis vanishes if

$$l^2 + l - 2 = 0 \quad \Rightarrow \quad l = 1 \text{ or } l = -2.$$

Since the equation is linear in Φ , any superposition of these solutions is valid, so we get:

$$f(r) = \begin{cases} K_1 r + K_2 r^{-2} & r < a, \\ C_1 r + C_2 r^{-2} & r > a. \end{cases}$$

Boundary conditions yield:

$$(42) \Rightarrow K_2 = 0,$$

$$(43) \Rightarrow C_1 = -E_0,$$

$$(40) \Rightarrow \varepsilon_1 K_1 - \varepsilon_0 (C_1 - 2C_2 a^3) = 0,$$

$$(41) \Rightarrow K_1 a - (C_1 a + C_2 a^{-2}) = 0.$$

The last two equations solve to:

$$K_1 = -3 \frac{\varepsilon_0}{2\varepsilon_0 + \varepsilon_1} E_0, \quad C_2 = \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} a^3 E_0,$$

which give as final answer:

$$f(r) = \begin{cases} -3 \frac{\varepsilon_0}{2\varepsilon_0 + \varepsilon_1} E_0 r & r < a, \\ -E_0 r + \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} \frac{a^3}{r^2} E_0 & r > a. \end{cases}$$

b) The potential we found is:

$$\Phi = \begin{cases} -3 \frac{\varepsilon_0}{2\varepsilon_0 + \varepsilon_1} E_0 z & r < a, \\ -E_0 z + \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} \frac{a^3}{r^3} E_0 z & r > a. \end{cases}$$

At $r > a$, we found the term from the external field $\mathbf{E}_0$, and the second term looks like a dipole potential, which means the polarized sphere produces a dipole field:

$$\frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} \frac{a^3}{r^3} E_0 z = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{d} \cdot \mathbf{x}}{r^3} \quad \text{for} \quad \mathbf{d} = \underbrace{4\pi\varepsilon_0 \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} a^3}_{\alpha} \mathbf{E}_0. \quad (44)$$

c) Imagine that the gas fills a sphere of radius R around the origin, with $r \gg R \gg n^{-1/3} \gg a$. This hierarchy of scales means the following. The typical distance $\lambda \sim n^{-1/3}$ between the small beads is much larger than their radius a , which means the beads don't interact (at the position of one bead, the dipole potential from an other bead scales like a^2/λ^2 which is negligible). But the sphere of radius R contains many beads, which means we can consider the gas homogeneous. And we look at that sphere from a very large distance

r . Then we compare the effect of this sphere of gas, to the effect we computed for a sphere of continuous material, to extract its effective permittivity ε .

From that distance, at first order every bead is located at the origin. Moreover, since they don't interact, they all get polarized like we computed before. So, we can simply add up their dipole contribution to the potential:

$$\begin{aligned}\Phi &= -E_0 z + \sum_i \frac{1}{4\pi\varepsilon_0} \frac{\alpha \mathbf{E}_0 \cdot (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3} \\ &\simeq -E_0 z + \frac{1}{4\pi\varepsilon_0} \frac{\alpha \mathbf{E}_0 \cdot \mathbf{x}}{r^3} \frac{4\pi R^3 n}{3} \\ &= -E_0 z + \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{d}' \cdot \mathbf{x}}{r^3}.\end{aligned}$$

Here the index i runs over all the beads. Like previously, we find the effect of the big sphere to amount to a dipole $\mathbf{d}' = \alpha' \mathbf{E}_0$, with

$$\alpha' = 4\pi\varepsilon_0 \left(\frac{4}{3} \pi n a^3 \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} \right) R^3 = 4\pi\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{2\varepsilon_0 + \varepsilon} \right) R^3.$$

The last equality defines ε , and is fixed by analogy with the definition of α in (44).

Thus, ε is given by

$$\frac{\varepsilon - \varepsilon_0}{2\varepsilon_0 + \varepsilon} = \frac{4}{3} \pi n a^3 \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1},$$

which yields:

$$\varepsilon = \varepsilon_0 \frac{1 + 2\xi}{1 - \xi}, \quad \xi = \frac{4}{3} \pi n a^3 \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1}.$$