

Classical Electrodynamics

Solutions Week 8

1. Consider the region $z > 0$ above an infinite conducting plane at $z = 0$.

- a) There is a charge q at the position $\mathbf{x}_1 = h \mathbf{e}_z$ and a charge $-q$ at the position $\mathbf{x}_2 = \mathbf{x}_1 - a \mathbf{e}_x$. Find the electrostatic potential Φ in the region $z > 0$.

In order to find the potential for $z > 0$, we use the method of the image charge: we put two charges in a symmetric position with respect to the initial ones, and we invert the charges. Therefore, we need to place a charge $-q$ in $\mathbf{x}'_1 = -h \mathbf{e}_z$ and a charge $+q$ in $\mathbf{x}'_2 = \mathbf{x}'_1 - a \mathbf{e}_x$ and the potential becomes

$$\Phi = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}_1|} - \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}_2|} - \left[\frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'_1|} - \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'_2|} \right]. \quad (1)$$

In order to check that our result makes sense, we can show that $\Phi = 0$ if $z = 0$. This is indeed the case, since for $z = 0$ we have

$$|\mathbf{x} - \mathbf{x}_1| = |\mathbf{x} - \mathbf{x}'_1|, \quad (2)$$

and

$$|\mathbf{x} - \mathbf{x}_2| = |\mathbf{x} - \mathbf{x}'_2|. \quad (3)$$

- b) There is a dipole $\mathbf{d} = d \mathbf{e}_x$ at a distance h from the conducting plane.

- i. Determine the electrostatic potential Φ in the region $z > 0$.

Hint: Use the previous question in the limit $a \rightarrow 0$ with $d = qa$ fixed.

- ii. Show that at large distances the potential is dominated by a quadrupole and determine the corresponding quadrupole tensor Q_{ij} .

Firstly, we notice that a charge q at the position $\mathbf{x}_1 = h \mathbf{e}_z$ and a charge $-q$ at the position $\mathbf{x}_2 = \mathbf{x}_1 - a \mathbf{e}_x$ gives rise to a dipole

$$\mathbf{d} = q \mathbf{x}_1 - q \mathbf{x}_2 = qa \mathbf{e}_x. \quad (4)$$

Therefore, in the limit $a \rightarrow 0$ with fixed product $qa = d$ the system of two charges reduces to a dipole $\mathbf{d} = d \mathbf{e}_x$ at the position $\mathbf{x}_1 = h \mathbf{e}_z$. In order to see what happens to the potential in this limit we rewrite equation (1) in the form

$$\Phi = \frac{q}{4\pi\epsilon_0} [f(x, y, z) - f(x + a, y, z)], \quad (5)$$

where

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + (z - h)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + h)^2}}. \quad (6)$$

Taylor expanding Φ around $a = 0$, we find the limit $a \rightarrow 0$ with fixed product $qa = d$ to be

$$\Phi \rightarrow -\frac{qa}{4\pi\epsilon_0} \partial_x f(x, y, z) \quad (7)$$

$$= \frac{dx}{4\pi\epsilon_0} \left[\frac{1}{(x^2 + y^2 + (z-h)^2)^{3/2}} - \frac{1}{(x^2 + y^2 + (z+h)^2)^{3/2}} \right]. \quad (8)$$

The final result is just the potential created by a dipole $\mathbf{d} = d\mathbf{e}_x$ at the position $\mathbf{x}_1 = h\mathbf{e}_z$ and an image dipole $-\mathbf{d}$ at the position $\mathbf{x}'_1 = -h\mathbf{e}_z$. At large distances $|\mathbf{x}|^2 = x^2 + y^2 + z^2$, we find

$$\Phi = \frac{dx}{4\pi\epsilon_0|\mathbf{x}|^3} \left[\left(1 + \frac{-2zh + h^2}{|\mathbf{x}|^2} \right)^{-3/2} - \left(1 + \frac{2zh + h^2}{|\mathbf{x}|^2} \right)^{-3/2} \right] \quad (9)$$

$$\approx \frac{6dhxz}{4\pi\epsilon_0|\mathbf{x}|^5} \quad (10)$$

Comparing with the quadrupole formula

$$\Phi = \frac{x^i Q_{ij} x^j}{8\pi\epsilon_0|\mathbf{x}|^5}, \quad (11)$$

we conclude that $Q_{xz} = Q_{zx} = 6dh$ and that all other components vanish. Notice that our result has the right dimensions of a quadrupole (charge times distance squared). Moreover, Q_{ij} is traceless as it should.

2. Antenna

A simple model of an antenna is given by the following current density:

$$\mathbf{J}(\mathbf{x}, t) = I \cos(\omega t) \Theta(a+z) \Theta(a-z) \delta(x) \delta(y) \mathbf{e}_z. \quad (12)$$

- a) Use the continuity equation to calculate the charge density $\rho(\mathbf{x}, t)$, assuming the initial condition $\rho(\mathbf{x}, 0) = 0$. Verify that the total charge is conserved.
- b) Calculate the total power radiated by the system. Recall that for this purpose it is sufficient to calculate the vector potential at very large distances $|\mathbf{x}| \gg \max(a, \lambda)$, where $\lambda = c/\omega$ is the wavelength of the emitted radiation. This is given by

$$\mathbf{A}(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \int d^3x' \mathbf{J} \left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}| + \frac{1}{c}\mathbf{n} \cdot \mathbf{x}' \right), \quad \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}. \quad (13)$$

Simplify your final result assuming that the source is *slow*: $\lambda \gg a$.

Solution

- a) The continuity equation gives:

$$\begin{aligned} \frac{\partial}{\partial t} \rho(\mathbf{x}, t) &= -\nabla \cdot \mathbf{J}(\mathbf{x}, t) = -\frac{\partial}{\partial z} J_z(\mathbf{x}, t) \\ &= -I \cos(\omega t) \delta(x) \delta(y) [\delta(z+a) \Theta(-z+a) - \delta(z-a) \Theta(z+a)] \\ &= -I \cos(\omega t) \delta(x) \delta(y) [\delta(z+a) - \delta(z-a)] \end{aligned}$$

Then the charge density is:

$$\rho(\mathbf{x}, t) = -\frac{I}{\omega} \sin(\omega t) \delta(x) \delta(y) [\delta(z+a) - \delta(z-a)] \quad (14)$$

The total charge is trivially conserved:

$$Q = \int_{\mathbb{R}^3} \rho(\mathbf{x}, t) dx dy dz = 0 \quad (15)$$

b) In the limit where $|\mathbf{x}| \gg \max(a, \lambda)$, the vector potential is given by:

$$\begin{aligned} \mathbf{A}(\mathbf{x}, t) &\simeq \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \int d^3x' \mathbf{J} \left(\mathbf{x}', t - \frac{1}{c} (|\mathbf{x}| - \mathbf{n} \cdot \mathbf{x}') \right), \quad \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} \\ &= \frac{\mu_0}{4\pi} \frac{1}{|\mathbf{x}|} \int d^3x' I \cos \left(\omega t - \frac{\omega}{c} (|\mathbf{x}| - \mathbf{n} \cdot \mathbf{x}') \right) \Theta(a+z') \Theta(a-z') \delta(x') \delta(y') \mathbf{e}_z \\ &= \frac{\mu_0}{4\pi} \frac{I}{|\mathbf{x}|} \int_{-a}^a dz' \cos \left(\omega t - \frac{\omega}{c} (|\mathbf{x}| - z' \cos \theta) \right) \mathbf{e}_z \\ &= \frac{\mu_0}{4\pi} \frac{I}{|\mathbf{x}|} \frac{c}{\omega \cos \theta} \left[\sin \left(\omega t - \frac{\omega}{c} (|\mathbf{x}| - a \cos \theta) \right) - \sin \left(\omega t - \frac{\omega}{c} (|\mathbf{x}| + a \cos \theta) \right) \right] \mathbf{e}_z \\ &= \frac{\mu_0}{2\pi} \frac{I}{|\mathbf{x}|} \frac{c}{\omega \cos \theta} \cos \left(\omega t - \frac{\omega |\mathbf{x}|}{c} \right) \sin \left(\frac{\omega a}{c} \cos \theta \right) \mathbf{e}_z \end{aligned}$$

Where in the integration from the second to the third line we aligned the azimuthal direction of the spherical coordinates (the direction pointing to the observer) with the vector $\mathbf{e}_z$.

Recalling that the field $\mathbf{B}$ is given by:

$$\mathbf{B} = \nabla \times \mathbf{A} \simeq \frac{\dot{\mathbf{A}} \times \mathbf{n}}{c} \quad (16)$$

we can compute the Poynting vector:

$$\begin{aligned} \mathbf{S}(\mathbf{x}, t) &\simeq \varepsilon_0 c^3 |\mathbf{B}|^2 \mathbf{e}_r = \varepsilon_0 c |\mathbf{e}_z \times \mathbf{n}|^2 \dot{A}_z^2 \mathbf{e}_r = \varepsilon_0 c \sin^2 \theta \dot{A}_z^2 \mathbf{e}_r \\ &= \varepsilon_0 c \sin^2 \theta \left[\frac{\mu_0}{2\pi} \frac{I}{|\mathbf{x}|} \frac{c}{\omega \cos \theta} \sin \left(\omega t - \frac{\omega |\mathbf{x}|}{c} \right) \sin \left(\frac{\omega a}{c} \cos \theta \right) \right]^2 \mathbf{e}_r \\ &= \frac{1}{4\pi^2 \varepsilon_0 c} \frac{I^2 \sin^2 \theta}{|\mathbf{x}|^2 \cos^2 \theta} \sin^2 \left(\omega t - \frac{\omega |\mathbf{x}|}{c} \right) \sin^2 \left(\frac{\omega a}{c} \cos \theta \right) \mathbf{e}_r \end{aligned}$$

The total power radiated by the system is obtained by computing the flux

of the time average of the Poynting vector on a sphere of large radius R :

$$\begin{aligned}
P &= \int \langle \mathbf{S}(\mathbf{x}, t) \rangle_t \mathbf{d}\sigma \\
&= \frac{1}{4\pi^2 \epsilon_0 c} \frac{I^2}{R^2} \underbrace{\left\langle \sin^2 \left(\omega t - \frac{\omega |\mathbf{x}|}{c} \right) \right\rangle_t}_{=1/2} \int_{-1}^1 \frac{\sin^2 \theta}{\cos^2 \theta} \sin^2 \left(\frac{\omega a}{c} \cos \theta \right) 2\pi R^2 d(\cos \theta) \\
&= \frac{I^2}{4\pi \epsilon_0 c} \int_{-1}^1 \frac{1-x^2}{x^2} \sin^2 \left(\frac{\omega a}{c} x \right) dx
\end{aligned}$$

In the limit where the source is slow ($\lambda \gg a$), a multipole expansion can be carried out. The first term contributing to the power is the dipole term:

$$P = \frac{I^2}{4\pi \epsilon_0 c} \int_{-1}^1 \frac{1-x^2}{x^2} \left(\frac{\omega a}{c} x \right)^2 dx = \frac{\omega^2 a^2 I^2}{3\pi \epsilon_0 c^3}. \quad (17)$$

3. Consider the vacuum region $z > 0$. At the surface $z = 0$, the electrostatic potential is given, $\Phi(x, y, 0) = \phi_0(x, y)$. Assume that $\phi_0(x, y)$ goes to zero rapidly when $r = \sqrt{x^2 + y^2} \rightarrow \infty$, or equivalently, that it has compact support.

a) Find an integral expression for the potential Φ in terms of the boundary data ϕ_0 .

Hint: Use an appropriate Green function as discussed in previous exercises.

b) Study the potential Φ for large values of $R = \sqrt{x^2 + y^2 + z^2}$. Show that the leading term in the large R expansion is the potential of a dipole. Write the dipole $\mathbf{d}$ in terms of ϕ_0 .

c) Assuming that the dipole $\mathbf{d} = 0$, show that the leading term has the form of a quadrupole. Determine the quadrupole in terms of ϕ_0 .

d) How do these results change if we consider the potential ϕ in the region $z > 0$ and $y > 0$, with boundary values ϕ_0 given on the two semi-planes ($z = 0 \wedge y > 0$ and $y = 0 \wedge z > 0$) that bound the region. What is the leading multipole (in general) in this case?

Solution

a) The electrostatic potential can be written as

$$\Phi(\mathbf{x}) = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x}', \mathbf{x}) d^3 \mathbf{x}' + \int_{\partial V} [G(\mathbf{x}', \mathbf{x}) \nabla \Phi(\mathbf{x}') - \Phi(\mathbf{x}') \nabla_{\mathbf{x}'} G(\mathbf{x}', \mathbf{x})] \cdot \mathbf{d}\sigma',$$

where G is a Green function. In this case, it is convenient to use a Green function obeying Dirichlet boundary conditions. The Green function is given

by (see problem 3 of Week 3)

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \right]. \quad (18)$$

In this case in which the boundary conditions is on the plane $z' = 0$, we have (don't forget that the surface vector points outside the surface so $\mathbf{d}\sigma' = -dx'dy'\mathbf{e}_{z'}$)

$$\nabla_{\mathbf{x}'} G \cdot d\sigma' = -\frac{\partial G}{\partial z'} dx'dy'. \quad (19)$$

The derivative of the Green function in $z' = 0$ is

$$\left. \frac{\partial G}{\partial z'} \right|_{z'=0} = \frac{1}{4\pi} \frac{2z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}. \quad (20)$$

Therefore, since there are no free charges, the potential can be written as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{2z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} \phi_0(x', y') dx'dy'. \quad (21)$$

b) The first term of an expansion for $R \gg R'$, where $R = |\mathbf{x}|$ and $R' = |\mathbf{x}'|$, is

$$\frac{1}{4\pi} \int \frac{2z}{R^3} \phi_0(x', y') dx'dy' = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{d} \cdot \mathbf{x}}{R^3}, \quad (22)$$

where we have defined

$$\mathbf{d} = \left(0, 0, 2\epsilon_0 \int \phi_0(x', y') dx'dy' \right). \quad (23)$$

As we can see, the leading term is a dipole.

c) If $\mathbf{d} = 0$, then one must expand further

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{6z(xx' + yy')}{R^5} \phi_0(x', y') dx'dy' + \dots \quad (24)$$

This corresponds to a quadrupolar potential

$$\Phi(\mathbf{x}) \simeq \frac{1}{4\pi\epsilon_0} \frac{1}{R^5} \frac{1}{2} Q_{ij} x_i x_j, \quad (25)$$

where the non-zero components of the quadrupole are

$$\begin{aligned} Q_{xz} = Q_{zx} &= 6\epsilon_0 \int \phi_0(x', y') x' dx'dy', \\ Q_{yz} = Q_{zy} &= 6\epsilon_0 \int \phi_0(x', y') y' dx'dy'. \end{aligned} \quad (26)$$

- d) In this case we need a Green function obeying Dirichlet boundary conditions on the semi-planes ($z = 0 \wedge y > 0$ and $y = 0 \wedge z > 0$). The Green function (18) can be thought of the potential coming from a charge placed in $\mathbf{x}'$ and an image charge opposite in sign, placed in $\mathbf{R}_z \mathbf{x}' = (x', y', -z')$. In this case, we can start by adding another image charge in $\mathbf{R}_y \mathbf{x}' = (x', -y', z')$. The charge in $\mathbf{x}'$ and in $\mathbf{R}_y \mathbf{x}' = (x', -y', z')$ now compensate on the semi-plane ($z > 0 \wedge y = 0$), but we still need to offset the potential from the charge placed in $\mathbf{R}_z \mathbf{x}'$. This can easily be done by adding one last image charge in $\mathbf{R}_z \mathbf{R}_y \mathbf{x}' = (x', -y', -z')$. The resulting Green function is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi} \left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y+y')^2 + (z-z')^2}} + \frac{1}{\sqrt{(x-x')^2 + (y+y')^2 + (z+z')^2}} \right]. \quad (27)$$

Notice that the arrangement and signs of the charges already imply that the leading term in the potential is a quadrupole (see exercise 2). Let us see this explicitly. We need the normal derivative of the Green function evaluated at the two boundaries:

$$-\left. \frac{dG}{dz'} \right|_{z'=0} = \frac{2z}{4\pi} \left[\frac{1}{((x-x')^2 + (y-y')^2 + z^2)^{3/2}} - \frac{1}{((x-x')^2 + (y+y')^2 + z^2)^{3/2}} \right] \quad (28)$$

$$-\left. \frac{dG}{dy'} \right|_{y'=0} = \frac{2y}{4\pi} \left[\frac{1}{((x-x')^2 + y^2 + (z-z')^2)^{3/2}} - \frac{1}{((x-x')^2 + y^2 + (z+z')^2)^{3/2}} \right] \quad (29)$$

The potential is

$$\begin{aligned} \Phi(\mathbf{x}) = & \frac{2z}{4\pi} \int_{-\infty}^{\infty} dx' \int_0^{\infty} dy' \left[\frac{1}{((x-x')^2 + (y-y')^2 + z^2)^{3/2}} - \frac{1}{((x-x')^2 + (y+y')^2 + z^2)^{3/2}} \right] \phi_0(x', y', 0) \\ & + \frac{2y}{4\pi} \int_{-\infty}^{\infty} dx' \int_0^{\infty} dz' \left[\frac{1}{((x-x')^2 + y^2 + (z-z')^2)^{3/2}} - \frac{1}{((x-x')^2 + y^2 + (z+z')^2)^{3/2}} \right] \phi_0(x', 0, z'). \end{aligned} \quad (30)$$

We now expand again for $R \gg R'$, keeping the first non zero contribution. The computation is analogous to the one in point c):

$$\begin{aligned} \Phi(\mathbf{x}) = & \frac{1}{4\pi R^5} \left(12zy \int_{-\infty}^{\infty} dx' \int_0^{\infty} dy' y' \phi_0(x', y', 0) \right. \\ & \left. + 12zy \int_{-\infty}^{\infty} dx' \int_0^{\infty} dz' z' \phi_0(x', 0, z') \right) + \dots \end{aligned} \quad (31)$$

Comparing with eq. (25), we find a quadrupole with the following non zero components:

$$Q_{yz} = Q_{zy} = 12\epsilon_0 \left(\int_{-\infty}^{\infty} dx' \int_0^{\infty} dy' y' \phi_0(x', y', 0) + \int_{-\infty}^{\infty} dx' \int_0^{\infty} dz' z' \phi_0(x', 0, z') \right). \quad (32)$$

4. We propose here a few questions to help you develop intuition about electrostatic multipoles.

- a) Find a charge distribution $\rho_1(\mathbf{x})$ which has monopole q but dipole $\mathbf{d} = \mathbf{0}$. You can take it as simple as possible, but what follows works for any such charge distribution.
- b) Consider the following charge distribution: $\rho_2(\mathbf{x}) = -\rho_1(\mathbf{x}) + \rho_1(\mathbf{x} - a\mathbf{e}_x)$. How does it look like ? Compute its monopole and dipole. Is the result surprising ?
- c) Now consider the charge distribution $\rho_3(\mathbf{x}) = \rho_1(\mathbf{x}) + \rho_2(\mathbf{x})$. How does it look like ? Compute its monopole and dipole. Is the result surprising ?
- d) Under what conditions are the monopole, the dipole or the quadrupole of a charge distribution invariant under translation of the charge density ? Can you generalize the result ?
- e) Find a charge distribution which has zero monopole, zero dipole, and with only non-zero quadrupole components $Q_{12} = Q_{21} \neq 0$. Do the same with only $Q_{23} = Q_{32} \neq 0$, and also with only $Q_{13} = Q_{31} \neq 0$. Can you have only $Q_{11} \neq 0$? Find one with only $Q_{11} = -Q_{22} \neq 0$, one with only $Q_{22} = -Q_{33} \neq 0$ and finally one with only $Q_{11} = -Q_{33} \neq 0$.
- f) If not already done, complete exercise 2. of last week.

Solution

- a) A very simple solution is given by a point charge q located at the origin: $\rho_1(\mathbf{x}) = q\delta^3(\mathbf{x})$. We will keep this solution to build the following steps. However, one could also do it with a more complicated solution, for example a homogeneous sphere $\rho_1'(\mathbf{x}) = \frac{q}{\frac{4}{3}\pi R^3}\Theta(R - |\mathbf{x}|)$.
- b) We consider $\rho_2(\mathbf{x}) = -q\delta^3(\mathbf{x}) + q\delta^3(\mathbf{x} - a\mathbf{e}_x)$, that is to say, one charge $-q$ at point $(0, 0, 0)$ and one charge q at point $(a, 0, 0)$. Clearly, this has monopole $Q = \int d^3x \rho_2(\mathbf{x}) = 0$, and dipole

$$Q_i = \int d^3x \rho_2(\mathbf{x}) x_i \Rightarrow Q_1 = qa, \quad Q_2 = Q_3 = 0. \quad (33)$$

This result is not surprising, as the distribution ρ_2 has total charge 0 and represents a physical dipole.

- c) Adding ρ_1 and ρ_2 , the two charges at the origin cancel and we get $\rho_3(\mathbf{x}) = q\delta^3(\mathbf{x} - a\mathbf{e}_x)$, just one charge q at position $(a, 0, 0)$. This is now the same distribution as in a), but translated by $a\mathbf{e}_x$. Computing the monopole gives as always the total charge $Q = q$ without surprise. The dipole gives:

$$Q_i = \int d^3x \rho_2(\mathbf{x})x_i \Rightarrow Q_1 = qa, \quad Q_2 = Q_3 = 0, \quad (34)$$

which is the same result as in question b). This is in fact not surprising if one refers to the superposition principle: since $\rho_3 = \rho_1 + \rho_2$, the multipoles of ρ_3 are the sum of the multipoles of ρ_1 and ρ_2 , because the formulas are linear in the charge distribution.

Now this may seem surprising since our terminology uses the word “dipole”, when the physical distribution is a single charge as in ρ_1 . The reason we get a non-zero dipole is that multipole moments are defined by an expansion around the origin. This choice of a particular point means that ρ_1 and ρ_2 are not equivalent, we don’t have translation invariance.

- d) To analyze the effect of translations, consider any distribution $\rho(\mathbf{x})$ and $\rho'(\mathbf{x}) = \rho(\mathbf{x} - \mathbf{x}_0)$, translated by $\mathbf{x}_0$. The monopole being the total charge, this of course never changes: $Q = \int d^3x \rho(\mathbf{x}) = Q' = \int d^3x \rho'(\mathbf{x})$. For the dipole:

$$\begin{aligned} Q'_i &= \int d^3x \rho'(\mathbf{x})x_i = \int d^3x \rho(\mathbf{x} - \mathbf{x}_0)x_i \\ &= \int d^3x \rho(\mathbf{x})(x + x_0)_i = \int d^3x \rho(\mathbf{x})(x_i + x_{0,i}) = Q_i + Qx_{0,i}. \end{aligned} \quad (35)$$

We did a simple change of variable in the third equality. We see that the difference between dipoles Q'_i and Q_i is the total charge times the translation vector, (as happened in c)). Thus, **the dipole of a charge distribution is translation-invariant if and only if the total charge (monopole) is zero**. Only in that case, like for example ρ_2 above, you can interpret the dipole term as telling you that your charge distribution resembles a physical dipole. Otherwise, the dipole is dependent of the choice of origin and only the monopole has a simple physical interpretation.

Similarly for the quadrupole:

$$\begin{aligned} Q'_{ij} &= \int d^3x \rho(\mathbf{x} - \mathbf{x}_0)(3x_ix_j - x^2\delta_{ij}) \\ &= \int d^3x \rho(\mathbf{x}) (3(x_i + x_{0,i})(x_j + x_{0,j}) - (x + x_0)^2\delta_{i,j}) \\ &= \int d^3x \rho(\mathbf{x}) ((3x_ix_j - x^2\delta_{ij}) + 3x_{0,i}x_j + 3x_{0,j}x_i + 3x_{0,i}x_{0,j} - (2x_{0,k}x_k + x_0^2)\delta_{ij}) \\ &= Q_{ij} + 3x_{0,i}Q_j + 3x_{0,j}Q_i + 3x_{0,i}x_{0,j}Q - 2\delta_{ij}x_{0,k}Q_k - \delta_{ij}x_0^2Q. \end{aligned} \quad (36)$$

This time the difference between Q'_{ij} and Q_{ij} is a combination of the monopole and dipole of the original distribution. This means **the quadrupole of a charge distribution is translation-invariant if and only if its monopole and dipole are zero**.

This reasoning easily generalizes to higher multipoles, implying that **only the first non-zero multipole of a charge distribution is translation-invariant and has an intuitive meaning.**

- e) Let us now find simple quadrupoles. You had an example in exercise 2 of last week. One can get a quadrupole by putting two opposite dipole next to each other. (In general you can imagine a 2^{n+1} -pole by putting two 2^n -poles next to each other). For example putting two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_x + a \mathbf{e}_y$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_x - a \mathbf{e}_y$ and $-\mathbf{x}_2$ produced a quadrupole $Q_{12} = Q_{21} = 12qa^2$ and all other components were 0 (and of course monopole and dipole were 0).

Other components can now be obtained by rotating this square of charges:

- two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_x + a \mathbf{e}_z$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_x - a \mathbf{e}_z$ and $-\mathbf{x}_2$ give only $Q_{13} = Q_{31} = 12qa^2$,
- two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_y + a \mathbf{e}_z$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_y - a \mathbf{e}_z$ and $-\mathbf{x}_2$ give only $Q_{23} = Q_{32} = 12qa^2$,
- two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_x$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_y$ and $-\mathbf{x}_2$ give only $Q_{11} = -Q_{22} = 6qa^2$,
- two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_y$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_z$ and $-\mathbf{x}_2$ give only $Q_{22} = -Q_{33} = 6qa^2$,
- two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_x$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_z$ and $-\mathbf{x}_2$ give only $Q_{11} = -Q_{33} = 6qa^2$.

Now you can build any quadrupole matrix by taking a linear combination of these results, thanks to superposition. Note that it is impossible that the only non-zero component is Q_{11} , because of the tracelessness property $Q_{ii} = 0$.