Classical Electrodynamics

Week 7

1. Calculate the leading behaviour of the electrostatic potential $\Phi(\mathbf{x})$ at large distances $|\mathbf{x}| \gg a$ for the following charge distributions:

a) One charge q at the point $\mathbf{x}_0 = a\,\mathbf{e}_x$ and one charge q at the point $-\mathbf{x}_0$.

b) One charge q at the point $\mathbf{x}_0 = a\,\mathbf{e}_x$ and one charge $-q$ at the point $-\mathbf{x}_0$.

c) Two charges q at the points $\mathbf{x}_1 = a\,\mathbf{e}_x + a\,\mathbf{e}_y$ and $\mathbf{x}_2 = a\,\mathbf{e}_x - a\,\mathbf{e}_y$ and two charges $-q$ at the points $-\mathbf{x}_1$ and $-\mathbf{x}_2$.

d) Two charges q at the points $\mathbf{x}_1 = a\,\mathbf{e}_x + a\,\mathbf{e}_y$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a\,\mathbf{e}_x - a\,\mathbf{e}_y$ and $-\mathbf{x}_2$.

e) Four charges q and four charges $-q$ placed at the eight vertices of a cube of side a, such that any two charges at distance a have opposite sign.

f) The following linear charge density along a ring of radius a:

$$\rho(r, \varphi, z) = \frac{q}{2\pi a} \cos(n\varphi)\,\delta(z)\delta(r-a)\,, \tag{1}$$

where $n = 0, 1, 2, \ldots$ and we used cylindrical coordinates (r, φ, z).

Solutions

a) The system has a total charge of $2q$, we can conclude that the leading behaviour is a monopole of charge $2q$.

We can also see this from the explicit form of the potential. The potential is:

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} + \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right]. \tag{2}$$

We define $R = \sqrt{x^2 + y^2 + z^2}$ and compute the leading term as $R \to \infty$:

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0 R} \left[\frac{1}{\sqrt{1 + \frac{-2xa+a^2}{R^2}}} + \frac{1}{\sqrt{1 + \frac{2xa+a^2}{R^2}}} \right] = \frac{2q}{4\pi\epsilon_0 R} + O\left(\frac{1}{R^2}\right).$$

We can read that the leading term is a monopole of charge $2q$.

b) Using the explicit form of the potential:

$$\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} \right], \tag{3}$$

we make the same expansion and get:

$$\begin{aligned}
\Phi(x,y,z) =& \frac{q}{4\pi\epsilon_0 R}\left[\frac{1}{\sqrt{1+\frac{-2xa+a^2}{R^2}}} - \frac{1}{\sqrt{1+\frac{2xa+a^2}{R^2}}}\right] \\
=& \frac{q}{4\pi\epsilon_0 R}\frac{2xa}{R^2} + O\left(\frac{1}{R^3}\right) = 2aq\frac{x}{4\pi\epsilon_0 R^3} + O\left(\frac{1}{R^3}\right) . \qquad (4)
\end{aligned}$$

We can read that the leading term is a dipole, of dipole moment $Q_1 = 2aq$, $Q_2 = 0$, $Q_3 = 0$.

Another way to treat the exercise is to use the explicit form of the multipole moments. The monopole is just the sum of all charges, it is zero. The dipole is given by the formula:

$$Q_i = \int \rho x_i d\mathbf{x} \qquad (5)$$

so $Q_2 = 0$, $Q_3 = 0$ and $Q_1 = qa + qa = 2qa$.

c) Here we use the explicit form of the multipole moments. The term of monopole is obviously zero, because the total charge is zero. Let's consider the term of dipole:

$$\mathbf{d} = qa\left(\mathbf{e}_x + \mathbf{e}_y\right) + qa\left(\mathbf{e}_x - \mathbf{e}_y\right) - qa\left(-\mathbf{e}_x - \mathbf{e}_y\right) - qa\left(-\mathbf{e}_x + \mathbf{e}_y\right) = 4qa\mathbf{e}_x \ . \qquad (6)$$

Since the term of dipole is different from zero, whereas the term of monopole is zero, the term of dipole is the leading term. The potential can thus be written as:

$$\Phi(\mathbf{x},t) = \frac{\mathbf{d}\cdot\mathbf{x}}{4\pi\epsilon_0 R^3} = \frac{qax}{\pi\epsilon_0 R^3} + O\left(\frac{1}{R^3}\right) \ . \qquad (7)$$

d) In this case, both the monopole and the dipole are zero. Indeed, the charges are the same as in the previous point, but they are positioned so that

$$\mathbf{d} = q\mathbf{x}_1 - q\mathbf{x}_1 - q\mathbf{x}_2 + q\mathbf{x}_2 = 0 \ . \qquad (8)$$

We must calculate the quadrupole and see if it is different from zero. The formula for the quadrupole is

$$Q_{ij} = \int d^3\mathbf{x}'\left(3x'_i x'_j - x'^2\delta_{ij}\right)\rho\left(\mathbf{x}'\right) \ , \qquad (9)$$

and the density is

$$\begin{aligned}
\rho\left(\mathbf{x}'\right) =& q\delta(z')\left(\delta\left(x'-a\right)\delta\left(y'-a\right) + \delta\left(x'+a\right)\delta\left(y'+a\right)\right) + \\
& - q\delta(z')\left(\delta\left(x'-a\right)\delta\left(y'+a\right) + \delta\left(x'+a\right)\delta\left(y'-a\right)\right) \ . \qquad (10)
\end{aligned}$$

Once we evaluate the integral, we obtain

$$\begin{aligned}
&Q_{xy} = Q_{yx} = 12qa^2 \, , \\
&Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0 \, , \\
&Q_{xx} = Q_{yy} = Q_{zz} = 0 \qquad (11)
\end{aligned}$$

Note that $\mathrm{Tr}\,(Q) = 0$, as we should expect. In an approximation up to the leading term, the potential is

$$\Phi(\mathbf{x}, t) = \frac{\mathbf{x}Q\mathbf{x}}{8\pi\epsilon_0 R^5} = \frac{12qa^2xy}{4\pi\epsilon_0 R^5} + O\left(\frac{1}{R^4}\right) \,. \tag{12}$$

e) If we place a positive charge at the origin $(0, 0, 0)$, we have three more positive charges at coordinates $(0, a, a)$, $(a, 0, a)$ and $(a, a, 0)$ while negative charges are sitting at $(0, 0, a)$, $(0, a, 0)$, $(a, 0, 0)$ and (a, a, a).

The potential is given by:

$$\begin{aligned}\Phi(x, y, z) = \frac{q}{4\pi\epsilon_0}\Bigg[&\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x^2+(y-a)^2+z^2}} \\ &+ \frac{1}{\sqrt{(x-a)^2+(y-a)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2}} \\ &+ \frac{1}{\sqrt{x^2+(y-a)^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z-a)^2}} \\ &+ \frac{1}{\sqrt{(x-a)^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{(x-a)^2+(y-a)^2+(z-a)^2}}\Bigg]\end{aligned} \tag{13}$$

We now expand in the big R limit. When the dust clears, we find :

$$\Phi(x, y, z) = \frac{-15q}{4\pi\epsilon_0 R^7} xyza^3 + O\left(\frac{1}{R^5}\right). \tag{14}$$

The leading term is an octupole, with only one non-zero moment (and its permutations) $Q_{123} = -15qa^3$. The octupole can also be computed directly by the formula

$$Q_{ijk} = \int d^3\mathbf{x}' \left(15x'_i x'_j x'_k - 3(\delta_{ij}x'_k + \delta_{ik}x'_j + \delta_{jk}x'_i)\mathbf{x}'^2\right)\rho(\mathbf{x}'). \tag{15}$$

f) In cylindrical coordinates we have

$$\mathbf{x} = (r\cos\varphi, r\sin\varphi, z) \,, \tag{16}$$

and, when $r' = a$ and $z' = 0$,

$$\mathbf{x}' = (a\cos\varphi', a\sin\varphi', 0) \,. \tag{17}$$

As a consequence, the potential can be written as

$$\begin{aligned}\Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{x}' \frac{\rho(r', \varphi', z')}{|\mathbf{x}-\mathbf{x}'|} = \\ &= \frac{1}{4\pi\epsilon_0}\int d\varphi' a \frac{q}{2\pi a}\frac{\cos(n\varphi')}{\sqrt{R^2+a^2-2ra\cos(\varphi-\varphi')}} = \\ &= \frac{1}{4\pi\epsilon_0}\int d\varphi' \frac{q}{2\pi}\frac{\cos(n\varphi')}{R\sqrt{1+\frac{a^2}{R^2}-2\frac{a}{R}\frac{r}{R}\cos(\varphi-\varphi')}} \,,\end{aligned} \tag{18}$$

where $R^2 = r^2 + z^2$. When $a \ll R$, we have

$$\begin{aligned}
&\frac{1}{\sqrt{1+\frac{a^2}{R^2}-2\frac{a}{R}\frac{r}{R}\cos(\varphi-\varphi')}} = \\
&= \sum_{k=0}^{l} \mathcal{L}_k\left(\frac{r}{R}\cos(\varphi-\varphi')\right)\left(\frac{a}{R}\right)^k + O\left(\frac{a}{R}\right)^{l+1} = \qquad (19)\\
&= \sum_{k=0}^{l}\left[\frac{(2k-1)!!}{k!}\left(\frac{r}{R}\cos(\varphi-\varphi')\right)^k + \mathcal{L}_{k-1}\left(\frac{r}{R}\cos(\varphi-\varphi')\right)\right]\left(\frac{a}{R}\right)^k + O\left(\frac{a}{R}\right)^{l+1},
\end{aligned}$$

where with $\mathcal{L}_k(x)$ we indicate a polynomial of order k of x. In the second equality of (19) we have written explicitly the term with the largest power in the polynomial. If we now use

$$\int_0^{2\pi} e^{i(l-n)\phi} d\phi = \left[\frac{e^{i(l-n)\phi}}{i(l-n)}\right]_0^{2\pi} = 0\,, \tag{20}$$

if $l \neq n$ and

$$\int_0^{2\pi} e^{i(l-n)\phi} d\phi = \int_0^{2\pi} d\phi = 2\pi\,, \tag{21}$$

if $n = l$, we can see that, when $l \leq n$,

$$\int_0^{2\pi} d\varphi' \cos(n\varphi')\left[\cos(\varphi'-\varphi)\right]^l = \frac{2\pi}{2^l}\cos(l\varphi)\,\delta_{n,l}\ . \tag{22}$$

Using this relation, we can finally obtain the potential at the leading term:

$$\begin{aligned}
\Phi(\mathbf{x},t) &\simeq \frac{1}{4\pi\epsilon_0}\frac{(2n-1)!!}{n!}\frac{q}{2^n R}\left(\frac{ra}{R^2}\right)^n \cos(n\varphi) + O\left(\frac{1}{R^{2n+2}}\right)\\
&= \frac{q}{4\pi\epsilon_0}\frac{(2n-1)!!}{2^n n!}\frac{(ra)^n}{R^{2n+1}}\cos(n\varphi) + O\left(\frac{1}{R^{2n+2}}\right).
\end{aligned}$$

In conclusion, the leading term is a n-pole.

2. Consider a particle of charge q moving with constant velocity $\mathbf{v} = (0, 0, v)$ along the z-axis. In a previous exercise, we have found that the retarded potentials are given by

$$\Phi(t,\mathbf{x}) = \frac{q\gamma}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}}\,, \qquad \mathbf{A}(t,\mathbf{x}) = \frac{\mathbf{v}}{c^2}\Phi(t,\mathbf{x})\,, \tag{23}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor.

a) Starting from the potentials, determine the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

b) Write the electromagnetic fields in terms of the norm of the vector $\mathbf{R}_t$ (vector between the charge q and the observer at time t) and the angle $\theta = \sphericalangle(\mathbf{R}_t, \mathbf{v})$. Study the fields in the non-relativistic ($v \ll c$) and ultra-relativistic ($v \approx c$) limits.

Solution

a) The electric and magnetic fields are related to the potentials with the following formulas:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \ . \tag{24}$$

We obtain the following result:

$$\begin{aligned}
E_x &= \frac{q\gamma}{4\pi\epsilon_0}\frac{x}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} \, , \\
E_y &= \frac{q\gamma}{4\pi\epsilon_0}\frac{y}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} \, , \\
E_z &= \frac{q\gamma}{4\pi\epsilon_0}\frac{\gamma^2(z-vt)}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} - \frac{q\gamma}{4\pi\epsilon_0 c^2}\frac{\gamma^2 v^2(z-vt)}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} = \\
&= \frac{q\gamma}{4\pi\epsilon_0}\frac{(z-vt)}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} \, , \\
B_x &= -\frac{q\gamma}{4\pi\epsilon_0 c^2}\frac{vy}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} \, , \\
B_y &= \frac{q\gamma}{4\pi\epsilon_0 c^2}\frac{vx}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} \, , \\
B_z &= 0 \, .
\end{aligned} \tag{25}$$

We can write the result in a more synthetic way:

$$\mathbf{E} = \frac{q\gamma}{4\pi\epsilon_0}\frac{\mathbf{R}_t}{\left(x^2+y^2+\gamma^2(z-vt)^2\right)^{3/2}} \ ,$$
$$\mathbf{B} = \frac{1}{c^2}\mathbf{v}\times\mathbf{E} \ , \tag{26}$$

where we have used $\mathbf{R}_t = (x, y, z - vt)$.

b) We can write

$$x^2 + y^2 = R_t^2 - z_t^2 \ , \tag{27}$$

where $z_t = z - vt = R_t \cos\theta$. Therefore we have

$$\begin{aligned}
\mathbf{E} &= \frac{q\gamma}{4\pi\epsilon_0}\frac{\mathbf{R}_t}{\gamma^3\left(\frac{1}{\gamma^2}(x^2+y^2)+(z-vt)^2\right)^{3/2}} = \\
&= \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\mathbf{R}_t}{\left(\frac{1}{\gamma^2}(R_t^2-z_t^2)+z_t^2\right)^{3/2}} = \\
&= \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\mathbf{R}_t}{\left(\left(1-\frac{v^2}{c^2}\right)R_t^2+\frac{v^2}{c^2}z_t^2\right)^{3/2}} = \\
&= \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\mathbf{R}_t}{\left(\left(1-\frac{v^2}{c^2}\right)R_t^2+\frac{v^2}{c^2}R_t^2\cos^2\theta\right)^{3/2}} = \\
&= \frac{q}{4\pi\epsilon_0\gamma^2 R_t^3}\,\frac{\mathbf{R}_t}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \ .
\end{aligned} \tag{28}$$

When $\frac{v}{c} \ll 1$ we obtain the laws of Coulomb and Biot-Savart:

$$\mathbf{E} = \frac{q\mathbf{R}_t}{4\pi\epsilon_0 R_t^3}\,,$$
$$\mathbf{B} = \frac{q\mathbf{v}\times\mathbf{R}_t}{4\pi\epsilon_0 R_t^3 c^2}\,. \tag{29}$$

When $\frac{v}{c} \simeq 1$, the electric field is reduced in the direction of $\mathbf{v}$ of a factor γ^2 and increased in the direction perpendicular to $\mathbf{v}$ of a factor γ.

3. A linearly polarised electromagnetic wave travels along the z-axis. It propagates in vacuum for $z < 0$ until it reaches a conductor, with conductivity σ, that fills the space for $z > 0$. The electric field of the incident wave is given by

$$\mathbf{E}_{in} = \mathrm{Re}\left[E_0 e^{i(kz-\omega t)}\right]\mathbf{e}_x \tag{30}$$

where $k = \omega/c$.

a) Show that the local Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ and Maxwell equations imply that

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} - \nabla^2\mathbf{B} + \mu_0\sigma\frac{\partial\mathbf{B}}{\partial t} = 0\,, \tag{31}$$

inside the conductor. For simplicity, we have assumed that the conductor has the electric permittivity and magnetic permeability of the vacuum.

b) Use the last equation to derive the form of the magnetic field inside the conductor,

$$\mathbf{B} = \mathrm{Re}\left[A e^{i(k'z-\omega t)}\right]\mathbf{e}_y\,, \qquad z > 0\,, \tag{32}$$

where A is an integration constant and $k' = \sqrt{k^2 + i\omega\mu_0\sigma}$.

c) What is the form of the electric field inside the conductor? How deep inside the conductor does the wave penetrate?

d) The electric field outside the conductor ($z < 0$) is a superposition of the incident and the reflected wave,

$$\mathbf{E} = \mathrm{Re}\left[E_0 e^{ikz-i\omega t} + E_1 e^{-ikz-i\omega t}\right]\mathbf{e}_x\,, \qquad z < 0\,. \tag{33}$$

Determine E_1 and A imposing continuity of the electric field and its first derivative at the interface $z = 0$. Notice that these follow from Maxwell equations in the absence of surface charge density and surface current. Show that the fraction of incident power reflected by the conductor is given by

$$\left|\frac{k'-k}{k'+k}\right|^2\,. \tag{34}$$

e) Verify that the rest of the incident power is dissipated in the conductor. Recall that the power dissipated per unit volume is given by $\mathbf{J}\cdot\mathbf{E}$.

f) Discuss your previous results in the high and low frequency limits ($\omega \gg \sigma/\epsilon_0$ and $\omega \ll \sigma/\epsilon_0$). What is the relevant regime for light incident over aluminium?

Solution

a) We can start by taking a time derivative of the third Maxwell equation:

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + \nabla \times \frac{\partial \mathbf{E}}{\partial t} = 0. \tag{35}$$

Taking into account Ohm's law, the second Maxwell equation is

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} - \frac{\sigma}{\epsilon_0}\mathbf{E}. \tag{36}$$

Pluggling this into (35), we get

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + c^2 \nabla \times (\nabla \times \mathbf{B}) + \frac{\sigma}{\epsilon_0}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{37}$$

where we used again the third Maxwell equation to eliminate the electric field. The last step uses the following identity, which you can prove quite easily or google even more easily:

$$\nabla \times (\nabla \times \mathbf{F}) = -\nabla^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F}). \tag{38}$$

The divergence of the magnetic field vanishes by the last Maxwell equation, and we finally obtain

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - c^2 \nabla^2 \mathbf{B} + \frac{\sigma}{\epsilon_0}\frac{\partial \mathbf{B}}{\partial t} = 0. \tag{39}$$

b) It is important to notice that partial differential equations have many solutions, and only when the initial conditions have been properly specified a unique solution is singled out. Therefore, we shouldn't aim at solving eq. (39) in general, but start from an ansatz that matches our physical expectations. To do so, it is useful to write the *imcoming* magnetic field

$$\mathbf{B}_{in} = \frac{1}{c}\mathbf{n} \wedge \mathbf{E}_{in} = \frac{1}{c}\mathrm{Re}\left[E_0 e^{i(kz-\omega t)}\right]\mathbf{e}_y\,. \tag{40}$$

Thus, the incoming wave only propagates in the z direction, and there is no reason to expect any dependence on x and y to kick in. Therefore, we look for a solution of the kind

$$\mathbf{B} = \mathbf{B}(z,t). \tag{41}$$

The magnetic field of the incoming wave is polarized along the $\mathbf{e}_y$ axis, and eq. (39) does not mix different components. This is enough to conclude that

$$\mathbf{B} = B(z,t)\mathbf{e}_y. \tag{42}$$

In the absence of the conductor ($\sigma = 0$), eq. (39) is just our beloved wave equation. Furthermore, the electromagnetic field coming from $z < 0$ is a plane wave. Turning on a small σ will affect the shape of the wave, but not alter the solution dramatically. We are therefore motivated to look for a solution in the form of exponentials:[1]

$$B(z,t) = A\, e^{\alpha t + \beta z}. \tag{43}$$

Plugging the ansatz in eq. (39) we get

$$\alpha^2 - c^2\beta^2 + \frac{\sigma}{\epsilon_0}\alpha = 0, \tag{44}$$

while the value of the magnetic field at $z = 0$ fixes

$$\alpha = -i\omega. \tag{45}$$

This implies

$$\beta^2 = \frac{1}{c^2}\left(-\omega^2 - i\frac{\sigma}{\epsilon_0}\omega\right), \tag{46}$$

that is,

$$\beta = \pm i\sqrt{k^2 + i\sigma\mu_0\omega}. \tag{47}$$

We end up with a wave that propagates in direction z and another that propagates in direction $-z$. But we never sent a wave from $z = +\infty$ backwards, so we choose the plus sign in eq. (47). We might be tempted to fix also the integration constant A in eq. (43) by matching it to $\mathbf{E}_{in}$, but this would be wrong: Maxwell equations - or continuity of the fields at $z = 0$ - imply that part of the wave is reflected, so the complete solution includes a plane wave propagating backwards in the region $z < 0$. We shall address this fact in point **d)**.

The news now is that the wave number $k' = \sqrt{k^2 + i\sigma\mu_0\omega}$ has an imaginary part. We can then write the magnetic field as follows:

$$\mathbf{B} = |A| e^{-\operatorname{Im} k'\, z} \cos(\operatorname{Re} k'\, z - \omega t)\mathbf{e}_y. \tag{48}$$

If $\operatorname{Im} k' > 0$, the wave is damped in the conductor, and penetrates only up to a depth

$$z_{\max} \simeq \frac{1}{\operatorname{Im} k'}. \tag{49}$$

Let us confirm that this is the case and compute $\operatorname{Im} k'$. Let us define the adimensional ratio

$$r \equiv \frac{\omega\sigma\mu_0}{k^2} = \frac{\sigma}{\omega\epsilon_0}. \tag{50}$$

The complex parameter k' can be written as

$$k' = k\sqrt{1 + ir} = k\,(1 + r^2)^{\frac{1}{4}}\left(\cos\frac{\arctan r}{2} + i\sin\frac{\arctan r}{2}\right) \tag{51}$$

[1] We could get to this conclusion also by making a more conservative ansatz. Eq. (39) can be solved by separation of variables $B(z,t) = B_1(z)B_2(t)$. Taking into account the initial conditions immediately leads to a solution in terms of exponentials. One could object that a different ansatz, *i.e.* $B(z,t) = B_1(z) + B_2(t)$ would also be possible. But this is incompatible with the initial conditions. Indeed, the magnetic field associated to $\mathbf{E}_{in}$ is such that $\frac{\partial \mathbf{B}_{in}}{\partial z}\Big|_{z=0}$ is not constant in time.

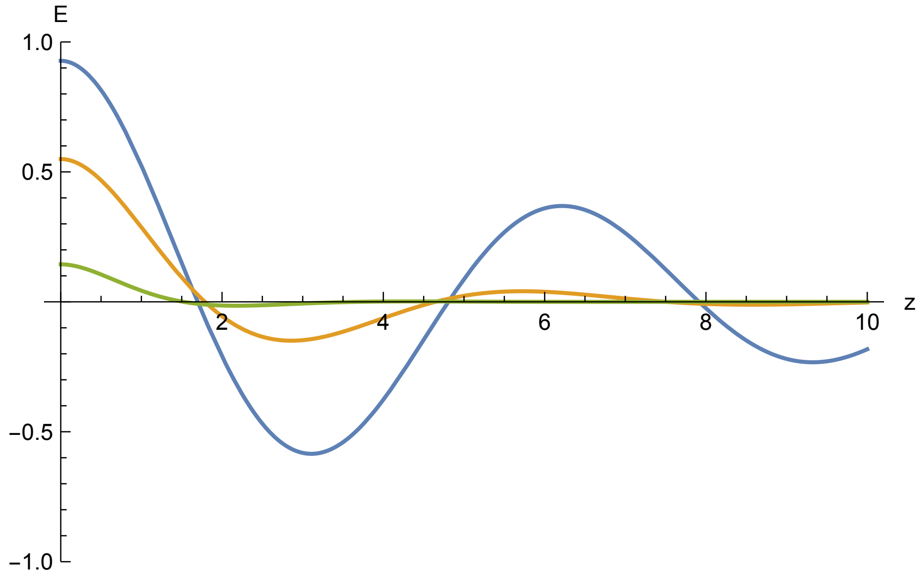

Figure 1: Plot of the electric field along the z direction for fixed time (arbitrary units) and different values of r: $r = 0.3$ (blue), $r = 1$ (orange) and $r = 3$ (green).

Remember that a square root defines two complex numbers, we chose the branch of the square root by demanding that $k' = k$ when $r = 0$. This confirms that

$$\operatorname{Im} k' = \operatorname{Im}(k\sqrt{1+ir}) = k\,(1+r^2)^{\frac{1}{4}} \sin\frac{\arctan r}{2} > 0. \tag{52}$$

Alternatively, we could have deduced the sign of $\operatorname{Im} k'$ by noticing that an exponentially growing energy towards positive z is unphysical.

c) As usual with waves, we deduce the electric field with the third Maxwell equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{53}$$

and knowing that $\mathbf{E}$ is only in the x direction, we get:

$$\partial_z E_x = \operatorname{Re}\left[i\omega A e^{i(k'z-\omega t)}\right], \tag{54}$$

so the electric field is (the integration constant is zero because we want a zero electric field in the limit $z \to \infty$):

$$\begin{aligned}\mathbf{E} =& \operatorname{Re}\left[\frac{\omega}{k'} A e^{i(k'z-\omega t)}\right]\mathbf{e}_x \\ =& \omega e^{-\operatorname{Im} k'\, z}\left[\operatorname{Re}\left(\frac{A}{k'}\right)\cos(\operatorname{Re} k'\, z - \omega t) - \operatorname{Im}\left(\frac{A}{k'}\right)\sin(\operatorname{Re} k'\, z - \omega t)\right]\mathbf{e}_x\,.\end{aligned} \tag{55}$$

This function is plotted in Figure (1).

d) We impose the continuity of $\mathbf{E}$ and $\partial_z \mathbf{E}$ at the interface $z = 0$. This translates in the complex system of equations:

$$\begin{cases} E_0 + E_1 = \frac{\omega A}{k'} \\ kE_0 - kE_1 = \omega A\,. \end{cases} \tag{56}$$

The solution of this system is:

$$\begin{cases} A = \frac{2E_0 k k'}{\omega(k+k')} = \frac{2E_0 k'}{c(k+k')} \\ E_1 = \frac{k-k'}{k+k'} E_0 \,. \end{cases} \tag{57}$$

Notice that while $\mathbf{B}$ is also continuous at the interface, $\partial_z \mathbf{B}$ is *not* because of Maxwell's second equation: $c^2 \nabla \times \mathbf{B} = \frac{\mathbf{J}}{\epsilon_0} + \partial_t \mathbf{E}$. However, $\mathbf{J} = 0$ for $z < 0$ and $\mathbf{J} = \sigma \mathbf{E}$ for $z > 0$.

Let us now compute the incident power of the wave (30). The Poynting vector is:

$$\mathbf{S} = \epsilon_0 c^2 \mathbf{E}_{in} \times \mathbf{B}_{in} = \epsilon_0 c |\mathbf{E}_{in}|^2 \mathbf{e}_z \,, \tag{58}$$

and writing the electric field explicitly,

$$\begin{aligned} |\mathbf{E}_{in}|^2 =& \operatorname{Re}\left[E_0 e^{i(kz-\omega t)}\right]^2 = (\operatorname{Re} E_0 \cos(kz-\omega t) - \operatorname{Im} E_0 \sin(kz-\omega t))^2 \\ =& (\operatorname{Re} E_0)^2 \cos^2(kz-\omega t) + (\operatorname{Im} E_0)^2 \sin^2(kz-\omega t) \\ &- 2 \operatorname{Re} E_0 \operatorname{Im} E_0 \cos(kz-\omega t) \sin(kz-\omega t) \,. \end{aligned} \tag{59}$$

The (average) incident power is the time-averaged flux of the Poynting vector through the surface $z = 0$. The sine cosine product drops after the time-averaging and we get:

$$P_{in} = \langle \mathbf{S} \rangle \cdot \mathbf{e}_z = \frac{1}{2} \epsilon_0 c |E_0|^2 \,. \tag{60}$$

(Note that strictly speaking, we are considering a power per unit surface. The total power is the integral of P over the entire surface which is infinite because in this model the wave is infinite in space).

We can now conclude that the fraction of incident power reflected by the conductor is:

$$f = \frac{P_{out}}{P_{in}} = \frac{\epsilon_0 c |E_1|^2/2}{\epsilon_0 c |E_0|^2/2} = \left| \frac{k'-k}{k'+k} \right|^2 \,. \tag{61}$$

e) The power dissipated in the volume of the conductor is:

$$P_{dis} = \int_0^\infty dz \langle \mathbf{J} \cdot \mathbf{E} \rangle = \sigma \int_0^\infty dz \langle |\mathbf{E}|^2 \rangle \,, \tag{62}$$

and denoting by $c = \cos(\operatorname{Re} k' \, z - \omega t)$ and $s = \sin(\operatorname{Re} k' \, z - \omega t)$, we have:

$$\begin{aligned} \langle |\mathbf{E}|^2 \rangle =& \left\langle \omega^2 e^{-2 \operatorname{Im} k' \, z} \left[\operatorname{Re}\left(\frac{A}{k'}\right)^2 c^2 + \operatorname{Im}\left(\frac{A}{k'}\right)^2 s^2 - 2 \operatorname{Re}\left(\frac{A}{k'}\right) \operatorname{Im}\left(\frac{A}{k'}\right) cs \right] \right\rangle \\ =& \omega^2 e^{-2 \operatorname{Im} k' \, z} \frac{1}{2} \left| \frac{A}{k'} \right|^2 \,. \end{aligned} \tag{63}$$

We get in the end:

$$P_{dis} = \sigma \frac{\omega^2}{4 \operatorname{Im} k'} \left| \frac{A}{k'} \right|^2 = \frac{\omega^2}{c^2} \frac{\sigma}{\operatorname{Im} k'} \frac{1}{|k+k'|^2} |E_0|^2 \,. \tag{64}$$

To see that is is equal to the non-reflected power, we need a bit of algebra:

$$\begin{aligned}\frac{P_{dis}}{P_{in}} =& 2\frac{\omega^2}{\epsilon_0 c^3}\frac{\sigma}{\operatorname{Im} k'}\frac{1}{|k+k'|^2} = 2\frac{\omega^2\mu_0\sigma}{c}\frac{1}{\operatorname{Im} k'}\frac{1}{|k+k'|^2} \\ =& \frac{2k\operatorname{Im}(k'^2)}{\operatorname{Im} k'}\frac{1}{|k+k'|^2} = \frac{4k\operatorname{Re}(k')\operatorname{Im}(k')}{\operatorname{Im} k'}\frac{1}{|k+k'|^2} = \frac{4k\operatorname{Re}(k')}{|k+k'|^2} \\ =& 1 - \left|\frac{k'-k}{k'+k}\right|^2 .\end{aligned} \tag{65}$$

The last step is easier to work out backwards:

$$\begin{aligned}1 - \left|\frac{k'-k}{k'+k}\right|^2 =& \frac{|k'+k|^2 - |k'-k|^2}{|k'+k|^2} = \frac{(k'+k)(k'+k)^* - (k'-k)(k'-k)^*}{|k'+k|^2} \\ =& \frac{2k(k'+k'^*)}{|k+k'|^2} = \frac{4k\operatorname{Re}(k')}{|k+k'|^2} .\end{aligned} \tag{66}$$

We can conclude that energy conservation is saved and everyone is happy!

f) In the low frequency regime,

$$\operatorname{Im} k' \simeq k\sqrt{\frac{r}{2}} = \sqrt{\frac{\omega\sigma\mu_0}{2}}, \qquad r \gg 1. \tag{67}$$

In the high frequency regime,

$$\operatorname{Im} k' \simeq \frac{1}{2}k\,r = \frac{\sigma}{2\,c\,\epsilon_0}, \qquad r \ll 1. \tag{68}$$

In fact, in real materials the conductivity depends on the frequency and the high frequency limit cannot be calculated this way.

In the case of an aluminium surface, $\sigma = 3.5\times10^7$ S/m so $\frac{\sigma}{\epsilon_0} = 4.0\times10^{18}\,\mathrm{s}^{-1}$. This corresponds to a wavelength of $\lambda = 2\pi\frac{c}{\omega} = 0.48$ nm. This distance is of the order of magnitude of the spacing between individual atoms, so in the context of classical electrodynamics, we are always in the large wavelength limit, i.e. the *low frequency* limit. The aluminium expels the EM fields and is a good reflector.