

Classical Electrodynamics

Week 6

1. Consider a particle of charge q moving with constant velocity $\mathbf{v} = (0, 0, v)$ along the z -axis. Show that the retarded potentials are given by

$$\Phi(t, \mathbf{x}) = \frac{q\gamma}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}} , \quad \mathbf{A}(t, \mathbf{x}) = \frac{\mathbf{v}}{c^2} \Phi(t, \mathbf{x}) , \quad (1)$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor.

Solution

We are going to use the formula

$$\Phi(\mathbf{x}, t) = \frac{1}{\epsilon_0} \int d^3\mathbf{x}' dt' G(\mathbf{x}', t', \mathbf{x}, t) \rho(\mathbf{x}', t') . \quad (2)$$

For a point-like particle moving in the direction of the z -axis, the density is

$$\rho(\mathbf{x}', t') = q \delta(x') \delta(y') \delta(z' - vt') . \quad (3)$$

As we have seen, the Green's function in this case is

$$G(\mathbf{x}', t', \mathbf{x}, t) = \frac{\delta(t - t' - \frac{|\mathbf{x}' - \mathbf{x}|}{c})}{4\pi|\mathbf{x}' - \mathbf{x}|} . \quad (4)$$

We may proceed in two different ways.

First method

We start with an integration with respect to t' :

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{x}' \frac{q \delta(x') \delta(y') \delta(z' - v(t - \frac{|\mathbf{x}' - \mathbf{x}|}{c}))}{|\mathbf{x}' - \mathbf{x}|} = \\ &= \frac{1}{4\pi\epsilon_0} \int dz' \frac{q \delta\left(z' - v\left(t - \frac{|(0, 0, z') - \mathbf{x}|}{c}\right)\right)}{|(0, 0, z') - \mathbf{x}|} . \end{aligned} \quad (5)$$

In equation (5), we have integrated over t' and then over x' and y' . We are left with an expression which contains a term of the form

$$\delta[f(z)] . \quad (6)$$

In a derivation that has been made in a previous exercise section, we found that the following equation holds

$$\delta[f(z)] = \sum_i \frac{\delta(z - z_{0i})}{|f'(z_{0i})|} , \quad (7)$$

where z_{0i} are defined as the points such that $f(z_{0i}) = 0$. We need to find the z'_0 for which we have

$$f(z'_0) = z'_0 - v \left(t - \frac{|(0, 0, z'_0) - \mathbf{x}|}{c} \right) = 0 . \quad (8)$$

In order to do so, we must solve a quadratic equation:

$$\begin{aligned}
vt - z'_0 &= \frac{v}{c} \sqrt{x^2 + y^2 + (z'_0 - z)^2} \\
(vt - z'_0)^2 &= \frac{v^2}{c^2} (x^2 + y^2 + (z'_0 - z)^2) \\
(vt)^2 + z'_0{}^2 - 2vtz'_0 &= \frac{v^2}{c^2} (z'_0{}^2 + r^2 - 2zz'_0) \\
z'_0{}^2(1 - \frac{v^2}{c^2}) - 2z'_0(vt - \frac{v^2}{c^2}z) - \frac{v^2}{c^2}r^2 + (vt)^2 &= 0 \\
z'_0 &= \gamma^2 \left[vt - \frac{v^2}{c^2}z - \sqrt{(vt - \frac{v^2}{c^2}z)^2 + \frac{1}{\gamma^2} \left(\frac{v^2}{c^2}r^2 - (vt)^2 \right)} \right] \\
z'_0 &= \gamma^2 \left[vt - \frac{v^2}{c^2}z - \frac{v}{c} \frac{1}{\gamma} \sqrt{x^2 + y^2 + \gamma^2(vt - z)^2} \right],
\end{aligned} \tag{9}$$

where we have defined $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Note that in (9) we have selected one of the two solutions of the quadratic equation. The other solution (which appeared because we took the square of the equation at some point) would not satisfy the first equation in (9), since it would have a negative term on the left hand side for some values of t (you may check this fact for $\mathbf{x} = 0$). For an interpretation of this see figure 1.

Let us calculate the derivative of f :

$$f'(z'_0) = 1 + \frac{v}{c} \frac{z'_0 - z}{|(0, 0, z'_0) - \mathbf{x}|}. \tag{10}$$

Moreover, using (8), we have

$$|(0, 0, z'_0) - \mathbf{x}| = ct - \frac{c}{v} z'_0. \tag{11}$$

Now we have everything we need to solve the integral in (5):

$$\begin{aligned}
\Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \frac{q}{|(0, 0, z'_0) - \mathbf{x}|} \frac{1}{|f'(z'_0)|} = \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{|(0, 0, z'_0) - \mathbf{x}|} \frac{1}{1 + \frac{v}{c} \frac{z'_0 - z}{|(0, 0, z'_0) - \mathbf{x}|}} = \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{|(0, 0, z'_0) - \mathbf{x}| + \frac{v}{c}(z'_0 - z)} = \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{ct - \frac{c}{v}z'_0 + \frac{v}{c}(z'_0 - z)} = \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{\frac{c}{v} \left(vt - \frac{z'_0}{\gamma^2} - \frac{v^2}{c^2}z \right)} = \\
&= \frac{1}{4\pi\epsilon_0} \frac{q}{\frac{c}{v} \left(\frac{v}{c} \frac{1}{\gamma} \sqrt{x^2 + y^2 + \gamma^2(vt - z)^2} \right)} = \\
&= \frac{1}{4\pi\epsilon_0} \frac{q\gamma}{\sqrt{x^2 + y^2 + \gamma^2(vt - z)^2}}.
\end{aligned} \tag{12}$$

Second method

Alternatively, we may decide to start with an integration with respect to $\mathbf{x}'$. In this case, the integral for the potential becomes

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int dt' \frac{q \delta \left[t' - \left(t - \frac{|(0,0,vt')-\mathbf{x}|}{c} \right) \right]}{|(0,0,vt')-\mathbf{x}|} . \quad (13)$$

If we call t'_0 the value such that

$$g(t'_0) = t'_0 - \left(t - \frac{|(0,0,vt'_0)-\mathbf{x}|}{c} \right) = 0 , \quad (14)$$

we can solve the integral in a way that is similar to the one of the first method. In this case

$$\frac{dg}{dt'}(t'_0) = 1 + \frac{v}{c} \frac{vt'_0 - z}{|(0,0,vt'_0)-\mathbf{x}|} . \quad (15)$$

Therefore, we can write the potential in the following way:

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\epsilon_0} \frac{q}{|(0,0,vt'_0)-\mathbf{x}|} \frac{1}{1 + \frac{v}{c} \frac{vt'_0 - z}{|(0,0,vt'_0)-\mathbf{x}|}} = \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{|(0,0,vt'_0)-\mathbf{x}| + \frac{v}{c}(vt'_0 - z)} = \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{c(t - t'_0) + \frac{v}{c}(vt'_0 - z)} = \\ &= \frac{1}{4\pi\epsilon_0} \frac{q}{ct - \frac{v}{c}z - ct'_0 \left(1 - \frac{v^2}{c^2}\right)} . \end{aligned} \quad (16)$$

Now we just need to find t'_0 and the solution will be complete. This is done by solving equation (14). The solution is

$$t'_0 = \gamma^2 \left[t - \frac{v}{c^2}z - \frac{1}{c} \sqrt{(z - vt)^2 + \frac{1}{\gamma^2}(x^2 + y^2)} \right] . \quad (17)$$

If we plug this solution into (16), we obtain

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \frac{q \gamma}{\sqrt{x^2 + y^2 + \gamma^2(vt - z)^2}} . \quad (18)$$

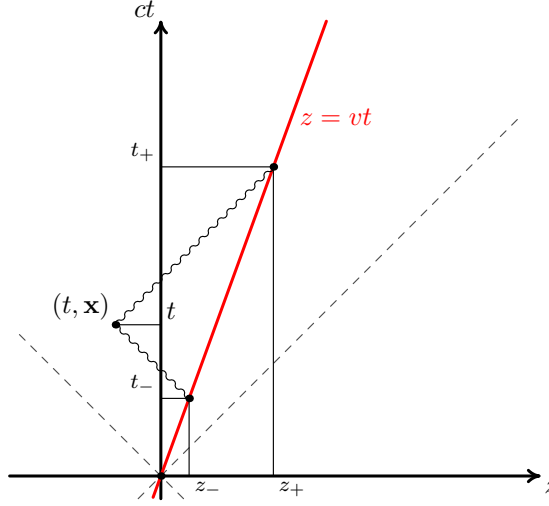


Figure 1: Depiction of the two solutions of the quadratic equation in (9). The wiggly line coming from below represents a *retarded* potential because $\Phi(t, \mathbf{x})$ depends on the position of the particle at a time t' before t , while the wiggly line coming from above represents an unphysical *advanced* potential as $\Phi(t, \mathbf{x})$ would depend on the future position of the particle.

2. Two infinitely-long grounded metal plates, located at $y = 0$ and at $y = a$, are connected at $x = \pm b$ by metal strips (again infinitely-long) maintained at a constant potential V_0 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe, by using the separation of variable method.

Solution

The coordinate system that looks best suited to solve this problem is the cartesian system, so we are going to adopt that in our solution. The configuration is independent of z , so $\Phi(x, y, z) \equiv \Phi(x, y)$ and the Laplace equation reduces to:

$$\frac{\partial \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial y^2} = 0, \quad (19)$$

with boundary conditions

$$\begin{cases} \Phi(x, y) \equiv 0 & \text{if } y = 0 \text{ or } y = a; \\ \Phi(x, y) \equiv V_0 & \text{if } x = \pm b. \end{cases}$$

We then proceed to the ansatz for variable separation, namely, we study the solutions of the form:

$$\Phi(x, y) = X(x)Y(y),$$

and by pasting it in equation (19) we get:

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0 \implies \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0,$$

where we “separated the variables” by dividing by Φ .

Now, we have reduced to an equation of the form $f(x) + g(y) = 0$, and the only possible solutions are those where both f and g are constants (deeply convince yourself of that by thinking of what would happen if you varied x without touching y).

Our problem has reduced then to the system:

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \tilde{C} \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\tilde{C};$$

in principle $\tilde{C}$ could take any value, but for reasons that will become evident in a moment, we will assume it to be positive, $\tilde{C} = k^2$. We are indeed left with two ordinary differential equations:

$$\frac{d^2 X}{dx^2} = k^2 X \quad \frac{d^2 Y}{dy^2} = -k^2 Y,$$

that have solutions:

$$X(x) = Ae^{kx} + Be^{-kx} \quad Y(y) = C \sin ky + D \cos ky$$

which gives a family of solutions labeled by k

$$\Phi_k(x, y) = (Ae^{kx} + Be^{-kx}) \cdot (C \sin ky + D \cos ky).$$

It remains to impose the right boundary conditions that will set the constants.

Here comes the crucial part of the variable separation method: the ansatz $\Phi(x, y) = X(x)Y(y)$ gives us an *infinite* family of solutions, *none* of them satisfying the boundary conditions by itself, but we can *linearly* combine them in a way that they do! This is possible because we are dealing with a complete set of functions, a basis for the full space of solutions.

Notice that our problem is symmetric in x , $\Phi(-x, y) = \Phi(x, y)$, which sets $A = B$, so

$$A = B \implies X(x) = A(e^{kx} + e^{-kx}) = 2A \cosh(kx).$$

We can reabsorb the constant A within C and D so to have

$$\Phi_k(x, y) = \cosh(kx) (C \sin ky + D \cos ky).$$

Let's impose the boundary condition on $y = 0, a$, we can do so for each solution Φ_k ¹:

$$\Phi_k(x, y = 0) = 0 \implies D = 0;$$

$$\Phi_k(x, y = a) = 0 \implies C \cdot \sin(ka) = 0 \implies k = \frac{n\pi}{a} \text{ with } n \in \mathbb{Z};$$

The general solution will then be a linear combination of the functions found above, so the potential can be written as:

$$\Phi(x, y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{a}x\right) \cdot \sin\left(\frac{n\pi}{a}y\right).$$

¹This is the reason why we chose $\tilde{C} > 0$, it becomes immediate to solve for the boundary conditions.

We can now impose the condition at $x = b$ (and then $x = -b$ will be automatically satisfied because we already imposed the symmetry of the solution under $x \rightarrow -x$). This reads:

$$\Phi(b, y) = \sum_{n=1}^{+\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \cdot \sin\left(\frac{n\pi}{a}y\right) = V_0.$$

The coefficient can be computed by means of some Fourier analysis, by remembering that $\sin y$ is an orthogonal complete basis on the set $y \in [0, \pi]$. In particular, the result is:

$$C_n \cosh\left(\frac{n\pi b}{a}\right) = \begin{cases} 0 & \text{if } n \text{ is even.} \\ \frac{4V_0}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

In conclusion, the final answer for the potential is:

$$\Phi(x, y) = \frac{4V_0}{\pi} \sum_{\text{odd } n=1}^{+\infty} \frac{1}{n} \frac{\cosh\left(\frac{n\pi}{a}x\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi}{a}y\right).$$

Note: This exercise corresponds to Example 3.4 in Griffiths's book.

3. The electric field associated with an electromagnetic wave travelling along the z -axis can be written as follows

$$\mathbf{E} = E_{0x} \cos(kz - \omega t + \theta) \mathbf{e}_x + E_{0y} \cos(kz - \omega t + \theta + \phi) \mathbf{e}_y \quad (20)$$

$$= \text{Re} \left[(J_x \mathbf{e}_x + J_y \mathbf{e}_y) \sqrt{E_{0x}^2 + E_{0y}^2} e^{i(kz - \omega t + \theta)} \right] \quad (21)$$

where the polarization of the wave is encoded in the two dimensional vector,

$$\mathbf{J} = (J_x, J_y) = \left(\frac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}}, \frac{E_{0y}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\phi} \right), \quad (22)$$

known as the Jones vector. Notice that $|\mathbf{J}|^2 = |J_x|^2 + |J_y|^2 = 1$.

Consider two electromagnetic waves $\mathbf{E}_1$ and $\mathbf{E}_2$ propagating along the z -axis with opposite circular polarisations and with the same frequency and phase.

- a) Find the Jones vector of each wave $\mathbf{E}_1$ and $\mathbf{E}_2$.
- b) Discuss, as a function of the amplitudes of each wave, the polarisation of the total wave and its associated Jones vector.

Solution

- a) The wave $\mathbf{E}_1$ has circular polarization. This means that when we fix z , for example $z = 0$, the electric field $\mathbf{E}_1(t)$ moves in a circle:

$$\begin{aligned} E_x(t) &= E_0 \cos(\omega t + \theta) \\ E_y(t) &= \pm E_0 \sin(\omega t + \theta). \end{aligned} \quad (23)$$

The sign in the second equation distinguishes the left and right polarization. $+$ is by convention the left-handed polarization and $-$ is the right handed polarization.

If $\mathbf{E}_1$ has left circular polarization, we see that we need $E_{1x} = E_{1y}$ and $\phi_1 = \frac{\pi}{2}$ in equation (22). This gives:

$$\begin{aligned}\mathbf{E}_1 &= E_1 \cos(kz - \omega t + \theta) \mathbf{e}_x - E_1 \sin(kz - \omega t + \theta) \mathbf{e}_y \\ \mathbf{J}_1 &= \frac{1}{\sqrt{2}} (1, i) .\end{aligned}\quad (24)$$

$\mathbf{E}_2$ has right circular polarization, which means that $\phi_2 = \frac{3\pi}{2}$ in equation (22). This gives a Jones vector:

$$\begin{aligned}\mathbf{E}_2 &= E_2 \cos(kz - \omega t + \theta) \mathbf{e}_x + E_2 \sin(kz - \omega t + \theta) \mathbf{e}_y \\ \mathbf{J}_2 &= \frac{1}{\sqrt{2}} (1, -i) .\end{aligned}\quad (25)$$

b) The total wave is $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. We can write:

$$\begin{aligned}\mathbf{E} &= (E_1 + E_2) \cos(kz - \omega t + \theta) \mathbf{e}_x + (E_2 - E_1) \sin(kz - \omega t + \theta) \mathbf{e}_y \\ &= \text{Re} \left[(J_x \mathbf{e}_x + J_y \mathbf{e}_y) \sqrt{(E_1 + E_2)^2 + (E_2 - E_1)^2} e^{i(kz - \omega t + \theta)} \right] .\end{aligned}\quad (26)$$

The total Jones vector is thus:

$$\mathbf{J} = \frac{1}{\sqrt{2} \sqrt{E_1^2 + E_2^2}} \begin{pmatrix} E_1 + E_2 \\ (E_1 - E_2)i \end{pmatrix} = \frac{1}{\sqrt{E_1^2 + E_2^2}} (E_1 \mathbf{J}_1 + E_2 \mathbf{J}_2) .\quad (27)$$

The polarization of the total wave is, depending on the cases:

- $E_1 = 0$ or $E_2 = 0$: left or right circular polarization.
- $E_1 = E_2$: linear polarization along x .
- $E_1 = -E_2$: linear polarization along y .
- $E_1 \neq E_2 \neq 0$: elliptic polarization.

4. The complex plane is a very useful tool to compute integrals. The Cauchy theorem

$$\oint_{\partial D} f(z) dz = 0 , \quad \text{if } f \text{ is analytic in } D ,\quad (28)$$

tells us that we can continuously deform the integration contour without changing the value of the integral (as long as we keep the endpoints fixed and do not cross any singularity). The residue theorem

$$\oint_{\partial D} f(z) dz = 2\pi i \sum_k \text{Res}(f, z_k) ,\quad (29)$$

reduces the contour integral calculation to the sum over residues of all poles of f inside the domain D (as long as f is single-valued inside D).

a) Use these ideas to calculate the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}, \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^4}. \quad (30)$$

b) Calculate also the Fourier transforms

$$\int \frac{d\omega}{2\pi} \frac{1}{1+\omega^2} e^{-i\omega t}, \quad \int \frac{d\omega}{2\pi} \frac{1}{\cosh \frac{\pi\omega}{2}} e^{-i\omega t}. \quad (31)$$

c) Use the same methods to find the form of the advanced and retarded Green functions in position space, starting from

$$G(\mathbf{x}, t) = \int \frac{d\omega d^3k}{(2\pi)^4} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} \frac{1}{\mathbf{k}^2 - \frac{1}{c^2}(\omega \pm i\epsilon)^2}, \quad (32)$$

where $\epsilon > 0$ is infinitesimal and the sign $\pm$ distinguishes the retarded from the advanced Green function.

Solution

a) (1) Consider the integral

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{+\infty} \frac{dx}{(x+i)(x-i)}. \quad (33)$$

In the complex plane, $f(z)$ has two simple poles, in $z = i$ and $z = -i$. In this particular case, we can close the path of integration in both the half planes, because in both cases $zf(z) \rightarrow 0$ when $|z| \rightarrow \infty$. Suppose we decide to close it in the upper plane, namely the plane with positive imaginary part. In that case, the only pole that is inside the contour of integration is $z = i$. Let γ_+ be a semicircle as represented in figure 2. Using the Residue theorem, we can write

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{\gamma_+} f(z) dz = 2\pi i \lim_{z \rightarrow i} \frac{1}{z+i} = \pi. \quad (34)$$

For pedagogical reasons, we will show that if we chose to close the contour in the other plane we would obtain the same result. This allow us to remind the reader of an important rule that must be respected when we integrate in the complex plane and we want to use the Residue theorem: the sign does not change if the path is followed keeping the inner part on the left, otherwise the sign needs to be changed. Therefore, if we call γ_- the circuit done in the lower half plane (figure 3), we have

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{\gamma_-} f(z) dz = -2\pi i \lim_{z \rightarrow -i} \frac{1}{z-i} = \pi. \quad (35)$$

(2) For the second integral, we start by solving the equation

$$z^4 = -1. \quad (36)$$

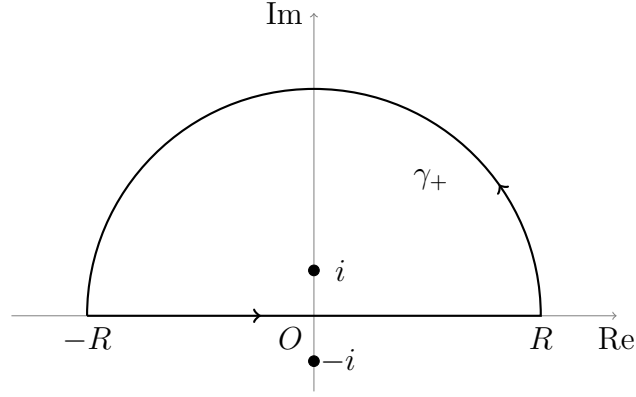


Figure 2: We close the contour in the upper plane. The inner part is kept on the left, if we follow the path.

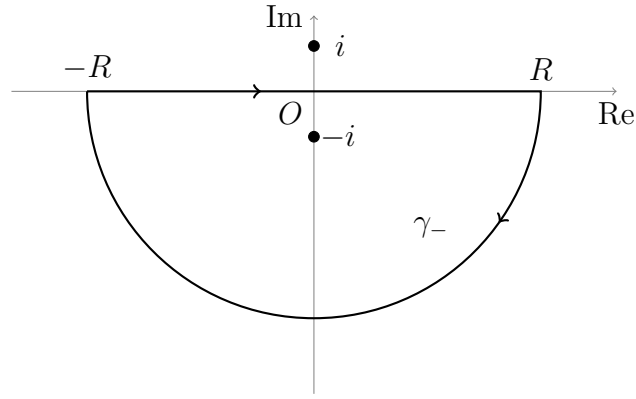


Figure 3: We close the contour in the lower plane. The inner part is kept on the right, if we follow the path: we must change the sign of the integral.

The solutions are

$$\begin{aligned}
 z_1 &= \frac{1}{\sqrt{2}}(1 + i) \\
 z_2 &= \frac{1}{\sqrt{2}}(1 - i) \\
 z_3 &= -\frac{1}{\sqrt{2}}(1 + i) \\
 z_4 &= -\frac{1}{\sqrt{2}}(1 - i) .
 \end{aligned} \tag{37}$$

Even in this case, we can choose to close the path in the upper or lower half plane. If we close it in the upper half plane, the poles are in $z = z_1$ and

$z = z_4$. We can proceed with the calculation:

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \frac{dx}{x^4 + 1} = \\
&= \int_{\gamma_+} \frac{dz}{z^4 + 1} = \\
&= \int_{\gamma_+} \frac{dz}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} = \\
&= 2\pi i \left[\lim_{z \rightarrow z_1} \frac{1}{(z - z_2)(z - z_3)(z - z_4)} + \lim_{z \rightarrow z_4} \frac{1}{(z - z_1)(z - z_2)(z - z_3)} \right] = \\
&= 2\pi i \left[\frac{1}{\sqrt{2}i\sqrt{2}(1+i)\sqrt{2}} + \frac{1}{(-\sqrt{2})(-\sqrt{2}(1-i))\sqrt{2}i} \right] = \\
&= \frac{\pi}{\sqrt{2}} \left[\frac{1}{(1+i)} + \frac{1}{(1-i)} \right] = \frac{\pi}{\sqrt{2}} .
\end{aligned} \tag{38}$$

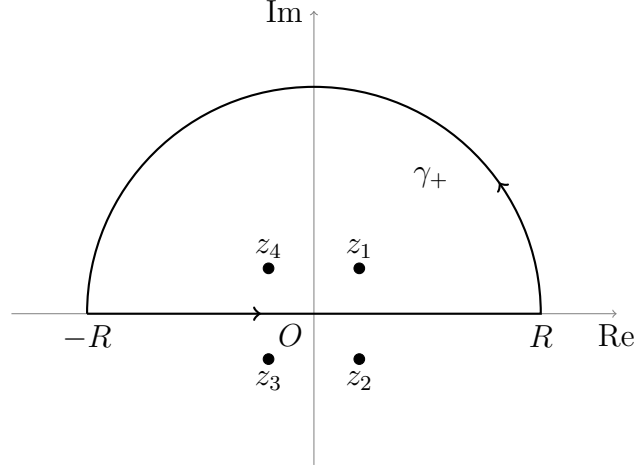


Figure 4: We close the contour in the upper plane.

b) (1)

$$I = \int \frac{d\omega}{2\pi} \frac{1}{1 + \omega^2} e^{-i\omega t} = \frac{1}{2\pi} \int \frac{d\omega}{(\omega + i)(\omega - i)} e^{-i\omega t} . \tag{39}$$

Now, if $t > 0$, we must close the path in the negative half of the imaginary axis, because we want the exponential to go to zero for $|\omega| \rightarrow \infty$. This is possible only if we can write ω as $\omega = a - ib$, with $b > 0$, where $a = \text{Re}[\omega]$ and $b = \text{Im}[\omega]$. In this way, we have $e^{-i\omega t} = e^{-(ia+b)t}$. Therefore the integral is

$$I = -2\pi i \lim_{\omega \rightarrow -i} \frac{1}{2\pi} \frac{1}{\omega - i} e^{-i\omega t} = \frac{e^{-t}}{2} . \tag{40}$$

If instead $t < 0$, we must close the path in the positive half. The integral becomes

$$I = 2\pi i \lim_{\omega \rightarrow i} \frac{1}{2\pi} \frac{1}{\omega + i} e^{-i\omega t} = \frac{e^t}{2} . \tag{41}$$

As a consequence the final result, which takes into account both the possibilities, is

$$I = \frac{e^{-|t|}}{2} . \quad (42)$$

(2)

$$I = \int \frac{d\omega}{2\pi} \frac{1}{\cosh \frac{\pi\omega}{2}} e^{-i\omega t} = \int \frac{d\omega}{2\pi} \frac{2}{\exp \left[\frac{\pi\omega}{2} \right] + \exp \left[\frac{-\pi\omega}{2} \right]} e^{-i\omega t} . \quad (43)$$

We must solve the equation

$$\exp \left[\frac{\pi\omega}{2} \right] = - \exp \left[\frac{-\pi\omega}{2} \right] = \exp \left[\frac{-\pi\omega}{2} \pm i(2k+1)\pi \right] , \quad (44)$$

where we have used

$$-1 = \exp[\pm i(2k+1)\pi] , \quad (45)$$

where k is a positive integer. The solution is

$$\omega = \pm i(2k+1) . \quad (46)$$

Therefore we have an infinite number of poles. Suppose first that $t > 0$. What we would like to do is to build a closed path in the complex plane with $\text{Im}[z] < 0$ and use the Residue theorem, summing the contributions that come from each pole. But are we allowed to do it?

In order to answer this question, we need to discuss the mathematical results that allow us to use the Residue theorem in the first place. When we have an integral of the form

$$\int_{-\infty}^{+\infty} f(x) dx , \quad (47)$$

what we usually do ² is to consider a closed half circle γ_R with radius R and to write

$$\int_{-R}^{+R} f(x) dx = \int_{\gamma_R} f(z) dz - \int_{\text{Arc}_R} f(z) dz , \quad (48)$$

where with Arc_R we indicate the half circle without the diameter. Then, we show that

$$\lim_{R \rightarrow \infty} \int_{\text{Arc}_R} f(z) dz = 0 , \quad (49)$$

and we are thus allowed to solve the integral with the Residue theorem. However, in this particular case, we cannot send R to infinity because every time R is an odd integer number the integral is not even defined, since the integrand is not analytical along the whole contour of integration. We thus impose $R = 2s$, where s is a positive integer, and we send s to infinity. If ω lies on a half circle of radius $2s$, we have, for every ω ,

$$\left| \frac{1}{2\pi} \frac{1}{\cosh \frac{\pi\omega}{2}} e^{-i\omega t} \right| \simeq \left| \frac{1}{2\pi} e^{-(t|\text{Im}[\omega]| + \frac{\pi}{2}|\text{Re}[\omega]|)} \right| \leq \frac{1}{2\pi} e^{-2a(t)s} , \quad (50)$$

²even if it has been left implicit in the previous integrals

with $a(t) > 0$ for every value of t . There is a theorem, called *estimation lemma*, which states that if

$$|f(z)| \leq M , \quad (51)$$

for all $z \in \gamma$, then

$$\left| \int_{\gamma} f(z) dz \right| \leq M l(\gamma) , \quad (52)$$

where $l(\gamma)$ is the length of γ . In our case, since $l(\text{Arc}_R) = 2\pi s$, we have, for $s \rightarrow \infty$,

$$\left| \int_{\text{Arc}_R} \frac{1}{2\pi} \frac{1}{\cosh \frac{\pi\omega}{2}} e^{-i\omega t} dz \right| \leq s e^{-2a(t)s} \rightarrow 0 . \quad (53)$$

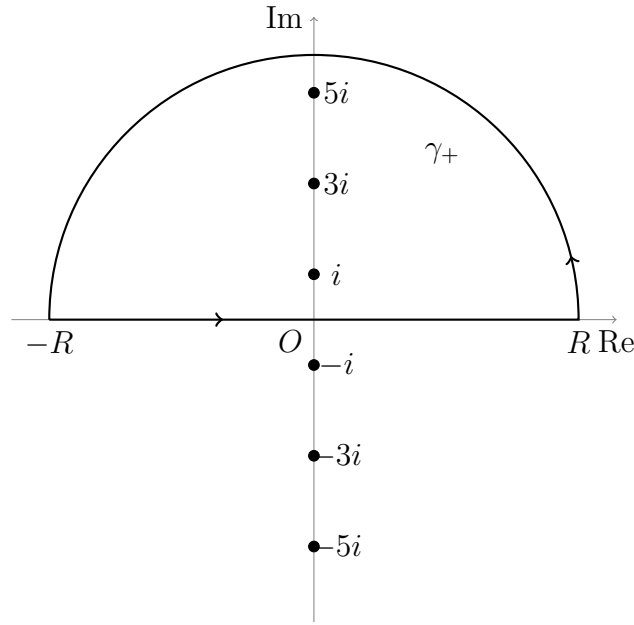


Figure 5: We close the contour in the upper plane.

Therefore the answer is yes, and we can proceed with the integral. As we said, $t > 0$ and we must close the contour of integration in the lower half plane. We expand the hyperbolic cosine around $\omega_k = -i(2k+1)$:

$$\cosh \frac{\pi\omega}{2} \simeq (-1)^{k+1} \frac{i\pi}{2} (\omega - \omega_k) . \quad (54)$$

We can use this result to solve the integral:

$$\begin{aligned} I &= -\frac{2\pi i}{2\pi} \sum_k \lim_{\omega \rightarrow \omega_k} \frac{e^{-i\omega t}}{(-1)^{k+1} \frac{i\pi}{2}} = \\ &= \sum_k \frac{2}{\pi} e^{-(2k+1)t} (-1)^k = \\ &= \frac{2}{\pi} e^{-t} \frac{1}{1 + e^{-2t}} = \frac{1}{\pi \cosh t} . \end{aligned} \quad (55)$$

If $t < 0$, the contour must be closed on the positive half plane and we write the poles as $\omega_k = i(2k + 1)$. The hyperbolic cosine in this case is

$$\cosh \frac{\pi\omega}{2} \simeq (-1)^k \frac{i\pi}{2} (\omega - \omega_k) , \quad (56)$$

and the integral is

$$\begin{aligned} I &= \frac{2\pi i}{2\pi} \sum_k \lim_{\omega \rightarrow \omega_k} \frac{e^{-i\omega t}}{(-1)^k \frac{i\pi}{2}} = \\ &= \sum_k \frac{2}{\pi} e^{-(2k+1)t} (-1)^k = \\ &= \frac{2}{\pi} e^{-t} \frac{1}{1 + e^{-2t}} = \frac{1}{\pi \cosh t} . \end{aligned} \quad (57)$$

c) $\mathbf{x}$ is a parameter of the integral. We first decompose $\mathbf{k}$ in spherical coordinates such that the azimuthal axis is aligned with $\mathbf{x}$ and so: $\mathbf{k} \cdot \mathbf{x} = kr \cos \theta$ where $r = |\mathbf{x}|$. Then the integral is:

$$\begin{aligned} G(\mathbf{x}, t) &= \int \frac{d\omega dk d\cos \theta d\varphi}{(2\pi)^4} k^2 e^{-i\omega t + ikr \cos \theta} \frac{1}{k^2 - \frac{1}{c^2}(\omega \pm i\epsilon)^2} \\ &= 2\pi \int \frac{d\omega dk d\cos \theta}{(2\pi)^4} k^2 e^{-i\omega t + ikr \cos \theta} \frac{1}{k^2 - \frac{1}{c^2}(\omega \pm i\epsilon)^2} \\ &= \int \frac{d\omega dk}{(2\pi)^3} \frac{k}{ir} e^{-i\omega t} [e^{ikr} - e^{-ikr}] \frac{1}{k^2 - \frac{1}{c^2}(\omega \pm i\epsilon)^2} \\ &= \int \frac{d\omega dk}{(2\pi)^3} \frac{k}{ir} e^{-i\omega t} [e^{ikr} - e^{-ikr}] \frac{-c^2}{(\omega \pm i\epsilon - ck)(\omega \pm i\epsilon + ck)} \\ &= \int \frac{d\omega dk}{(2\pi)^3} \frac{k}{ir} e^{-i\omega t} [e^{ikr} - e^{-ikr}] \frac{-c^2}{(\omega - \omega_1)(\omega - \omega_2)} , \end{aligned} \quad (58)$$

where we have defined

$$\begin{aligned} \omega_1 &= +ck \mp i\epsilon , \\ \omega_2 &= -ck \mp i\epsilon . \end{aligned} \quad (59)$$

We will now integrate with respect to ω . From now on, we will distinguish between the "+" and the "-" in equation (59).

If we choose

$$\begin{aligned} \omega_1 &= +ck + i\epsilon , \\ \omega_2 &= -ck + i\epsilon , \end{aligned} \quad (60)$$

the poles have positive imaginary part, therefore the integral is zero if $t > 0$, since we must close the contour in the lower half plane. If instead $t < 0$, we

have (sending ϵ to zero)

$$\begin{aligned}
G(\mathbf{x}, t) &= 2\pi i \int_0^\infty \frac{dk}{(2\pi)^3} \frac{k}{ir} [e^{ikx} - e^{-ikr}] [e^{-i\omega_1 t} - e^{-i\omega_2 t}] \frac{-c^2}{2ck} \\
&= \int_0^\infty \frac{dk}{(2\pi)^2} [e^{ikr} - e^{-ikr}] [e^{-i\omega_1 t} - e^{-i\omega_2 t}] \frac{-c}{2r} \\
&= \int_0^\infty \frac{dk}{(2\pi)^2} [e^{ikr} - e^{-ikr}] [e^{-i\omega_2 t} - e^{-i\omega_1 t}] \frac{c}{2r} \\
&= \frac{1}{2} \int_{-\infty}^\infty \frac{dk}{(2\pi)^2} [e^{ik(r+ct)} + e^{-ik(r+ct)} - e^{ik(r-ct)} - e^{-ik(r-ct)}] \frac{c}{2r} \\
&= \frac{c}{(4\pi)r} (\delta(r+ct) - \delta(r-ct)) \\
&= \frac{c}{4\pi} \frac{\delta(r+ct)}{r} .
\end{aligned} \tag{61}$$

In the last line, remember that $r \geq 0$ and $t < 0$ so only the first delta function is relevant. To conclude we have found:

$$G(\mathbf{x}, t) = \Theta(-t) \frac{c}{4\pi} \frac{\delta(|\mathbf{x}| + ct)}{|\mathbf{x}|} \tag{62}$$

On the contrary, if we choose

$$\begin{aligned}
\omega_1 &= +ck - i\epsilon , \\
\omega_2 &= -ck - i\epsilon ,
\end{aligned} \tag{63}$$

the poles have negative imaginary part and the integral is zero whenever $t < 0$. For $t > 0$, the calculation goes as before. In conclusion we have

$$G(\mathbf{x}, t) = \Theta(t) \frac{c}{4\pi} \frac{\delta(|\mathbf{x}| - ct)}{|\mathbf{x}|} . \tag{64}$$

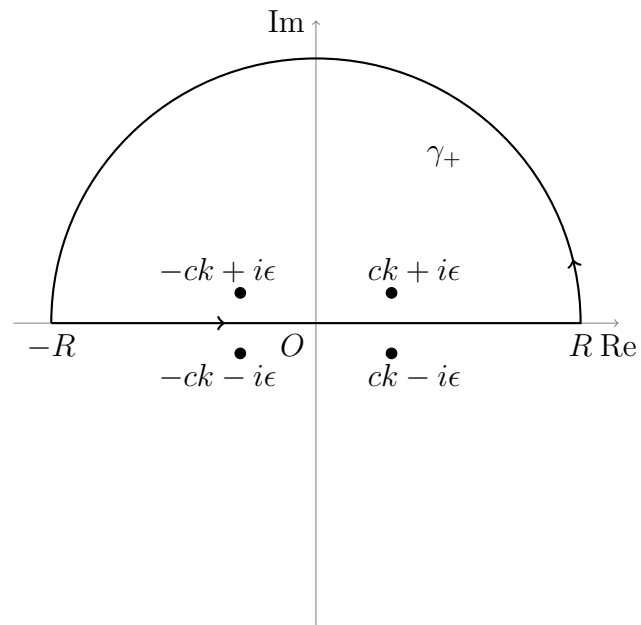


Figure 6: We close the contour in the upper plane.