Classical Electrodynamics

Solutions week 5

1. Consider the interior $\mathcal{V}$ of a spherical shell with radius R centred at point $\mathbf{x}_0$. There is no electric charge inside the shell and there is an arbitrary profile of electrostatic potential on the shell $\Phi(\mathbf{x})\big|_{|\mathbf{x}|=R} = V(\theta, \varphi)$.

a) Using the Green's identity for two scalar Φ and Ψ:

$$\int_{\mathcal{V}} d^3r\,(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi) = \oint_{\partial\mathcal{V}} \mathbf{dS}\cdot(\Phi\nabla\Psi - \Psi\nabla\Phi), \tag{1}$$

show that the potential at center $\mathbf{x}_0$ is the average of potential $V(\theta,\varphi)$ over the shell:

$$\Phi(\mathbf{x}_0) = \int_{\partial\mathcal{V}} V(\theta,\varphi)\,d\Omega\,, \tag{2}$$

with $d\Omega = \frac{1}{4\pi}\sin\theta\,d\theta\,d\varphi$.

Hint 1: Introduce a function Ψ that simplifies the l.h.s of equation (1) to the potential at center i.e. $\Phi(\mathbf{x}_0)$.

Hint 2: Use $\oint_{\partial\mathcal{V}} \mathbf{dS}\cdot\nabla\Phi(\mathbf{x}) = 0$ to simplify the r.h.s of equation (1).

b) Use this result to argue that maxima or minima of a solution to Laplace equation can only be at the boundary of the domain.

Solution

a) Choose the Φ in Green's identity to be the potential. We want to simplify the l.h.s. of Greens's identity to the potential at the center. First of all, the second term vanishes since there is no charge inside: $\nabla^2\Phi = -\rho/\varepsilon_0 = 0$. Furthermore, if we pick the Laplacian of Ψ to be a delta function at center $\nabla^2\Psi = \delta^3(\mathbf{x}-\mathbf{x}_0)$ then the l.h.s. would be the potential at center:

$$\int_{\mathcal{V}} d^3r\,\Phi\nabla^2\Psi = \int_{\mathcal{V}} d^3r\,\Phi\,\delta^3(\mathbf{x}-\mathbf{x}_0) = \Phi(\mathbf{x}_0). \tag{3}$$

As we have already seen before, the solution for $\nabla^2\Psi = \delta^3(\mathbf{x}-\mathbf{x}_0)$ is similar to the potential of charge density $\rho = q\delta^3(\mathbf{x}-\mathbf{x}_0)$ i.e. point-like charge:

$$\Psi(\mathbf{x}) = -\frac{1}{4\pi r}, \tag{4}$$

and therefore its gradient is similar to the electric field of a point-like charge:

$$\nabla\Psi(\mathbf{x}) = \frac{1}{4\pi r^2}\hat{\mathbf{r}}. \tag{5}$$

We can now compute the r.h.s. of Green's identity. The first term is:

$$\oint_{\partial\mathcal{V}} \mathbf{dS}\cdot\Phi\nabla\Psi = \oint_{\partial\mathcal{V}} \mathbf{dS}\cdot V(\theta,\varphi)\frac{\hat{\mathbf{r}}}{4\pi R^2} = \frac{1}{4\pi}\oint_{S^2} V(\theta,\varphi)d\Omega, \tag{6}$$

where we used the fact that $\mathbf{dS} = R^2 d\Omega\hat{\mathbf{r}}$, with $d\Omega = \sin\theta\, d\theta\, d\varphi$. The final integral is over a unit sphere S^2, and is indeed the average of the potential on the shell.

To complete the proof, we show the last term equals 0. The function Ψ is constant on $\partial\mathcal{V}$, and can be factored out of the integral. The remaining integral is zero because of Gauss' law:

$$\oint_{\partial\mathcal{V}} \mathbf{dS}\cdot\nabla\Phi(\mathbf{x}) = -\int_{\mathcal{V}}\frac{\rho(\mathbf{x})}{\varepsilon_0} = 0. \tag{7}$$

b) The Laplace equation $\nabla^2\Phi = 0$ corresponds to a potential in a region with no electric charge. Consider a (connected) domain $\mathcal{D}$ on which the function Φ solves the Laplace equation. We show that the maxima and minima of the solution Φ are located on the boundary $\partial\mathcal{D}$:

Proceed by contradiction: assume that some point $\mathbf{x}_0$ in the interior of $\mathcal{D}$ is the minimum of Φ on $\mathcal{D}$. Then, we can consider a ball centered at $\mathbf{x}_0$ and countained in the domain $\mathcal{D}$. By the property we showed above, the average of Φ on the surface of the sphere is equal to $\Phi(\mathbf{x}_0)$. Because of our assumption, this is only possible if $\Phi(\mathbf{x}) = \Phi(\mathbf{x}_0)$ on the shell. Therefore, the function is constant on the ball, and this reasoning can be extended to the whole domain $\mathcal{D}$. Hence Φ is a constant on the whole domain, and any point of the boundary is also a maximum. The same argument works for the minimum.

2. Consider a sphere of radius R with a fixed potential $\Phi(R,\theta,\varphi) = V(\theta,\varphi)$ on its surface.

a) Find an integral expression for the potential $\Phi(r,\theta,\varphi)$ in all space outside the sphere. **Hint:** Start by finding the appropriate Green function $G(\mathbf{r},\mathbf{r}')$ for this problem. Recall the general solution of Poisson equation

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r}',\mathbf{r})d^3\mathbf{r}' + \int_{\partial V}[G(\mathbf{r}',\mathbf{r})\nabla_{\mathbf{r}'}\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla_{\mathbf{r}'}G(\mathbf{r}',\mathbf{r})]\cdot\mathbf{d}\sigma'\,.$$

b) What is the leading behaviour of the potential $\Phi(r,\theta,\varphi)$ far away from the sphere, *i.e.* for $r \gg R$?

c) Consider now the following experiment. We take a metallic ball, cut it in half and glue it back together using an insulating glue. Then we establish a potential difference V_0 between the two half-balls (keeping the total charge of the ball equal to zero). Find an integral expression for the potential Φ outside the ball. How does Φ behave far away from the ball?

d) Estimate the total charge accumulated in each half-ball assuming that the thickness d of the insulating glue is much smaller than the radius of the ball.

e) Assume that the interior of the sphere is empty and the potential on the surface is given by $\Phi(R, \theta, \varphi) = V(\theta, \varphi)$. Find an integral expression for the potential $\Phi(r, \theta, \varphi)$ inside the sphere. What is the potential at the center of the sphere?

Solution

a) We will use the same method as the exercise 3 of the exercise sheet of Week 3. The potential Φ satisfies a Poisson equation with Dirichlet boundary conditions:

$$\begin{cases} -\nabla^2 \Phi = 0 \\ \Phi(R, \theta, \varphi) = V(\theta, \varphi). \end{cases} \tag{8}$$

The solution to this problem is given by the formula given in the text where the Green function G is chosen such that:

$$\begin{cases} -\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}') \\ G(\mathbf{r}, \mathbf{r}')\big|_{|\mathbf{r}|=R} = 0, \end{cases} \tag{9}$$

with boundary conditions of the Green function appropriate to solve a Dirichlet boundary condition problem. With this choice the first term in the boundary integral vanishes and the knowledge of Φ only in ∂V is sufficient to determine Φ everywhere.

The first step is to find the Green function. Remember that the Green function $G(\mathbf{r}, \mathbf{r}')$ is formally equivalent to the potential at $\mathbf{r}$ created by a point charge of charge unity placed at $\mathbf{r}'$. For this particular Green function, the volume V is the volume outside the sphere with boundary conditions $G(\mathbf{r}, \mathbf{r}')\big|_{|\mathbf{r}|=R} = 0$ (zero potential on the sphere). This problem has been solved in the third exercise of the first week problem sheet with the image charge method.

We recall that for a charge q placed at $\mathbf{r}'$, one considers an image charge in the sphere of charge q'' and position $\mathbf{r}''$ given by:

$$\begin{cases} q'' = -\frac{R}{r'} q \\ \mathbf{r}'' = \frac{R^2}{r'} \hat{r}' = \frac{R^2}{(r')^2} \mathbf{r}'. \end{cases} \tag{10}$$

Using the same technique, we can deduce the expression of the Green function satisfying (9):

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{R}{r'} \frac{1}{|\mathbf{r} - \mathbf{r}''|} \right]. \tag{11}$$

Let us express the Green function with the coordinates (r, θ, φ) and (r', θ', φ') of $\mathbf{r}$ and $\mathbf{r}'$ ($\mathbf{r}''$ has coordinates $(R^2/r', \theta, \varphi)$). We define γ the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$ and we have:

$$\begin{aligned}G(\mathbf{r},\mathbf{r}') &= \frac{1}{4\pi}\left[\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R}{r'}\frac{1}{\sqrt{r^2+\frac{R^4}{r'^2}-2\frac{rR^2}{r'}\cos\gamma}}\right] \\ &= \frac{1}{4\pi}\left[\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R}{\sqrt{r^2r'^2+R^4-2rr'R^2\cos\gamma}}\right].\end{aligned} \tag{12}$$

We have $\cos\gamma = \hat{r}\cdot\hat{r}'$ so plugging in coordinates, one gets:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'). \tag{13}$$

We have determined the Green function satisfying (9) (notice the symmetry between $\mathbf{r}$ and $\mathbf{r}'$), now we can plug the expression (12) in the formula of the general solution of Poisson equation given in the text. Since there is no volumic charge and the Green function has been chosen to vanish on the sphere, one has:

$$\Phi(\mathbf{r}) = -\int_{\partial V}\left[\Phi(\mathbf{r}')\nabla_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')\right]\cdot \mathbf{d}\sigma'\,. \tag{14}$$

$\mathbf{d}\sigma' = -R^2\sin\theta' d\theta' d\varphi' \hat{r}'$ (remember that the area vector is by convention pointing *outside* the volume) so the above formula becomes:

$$\Phi(\mathbf{r}) = R^2\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \left[V(\theta',\varphi')\left.\frac{\partial G}{\partial r'}\right|_{r'=R}\right], \tag{15}$$

and from the explicit formula (12), we get:

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=R} = \frac{1}{4\pi R}\frac{r^2-R^2}{(r^2+R^2-2rR\cos\gamma)^{\frac{3}{2}}}, \tag{16}$$

so we can conclude:

$$\Phi(\mathbf{r}) = \frac{R}{4\pi}\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \left[V(\theta',\varphi')\frac{r^2-R^2}{(r^2+R^2-2rR\cos\gamma)^{\frac{3}{2}}}\right]. \tag{17}$$

b) The value of $\cos\gamma$ does not change the leading behaviour when r goes to infinity.

$$\frac{r^2-R^2}{(r^2+R^2-2rR\cos\gamma)^{\frac{3}{2}}} \to \frac{r^2}{r^3}, \tag{18}$$

so:

$$\Phi(\mathbf{r}) \to \frac{R}{4\pi r}\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi'\, V(\theta',\varphi'). \tag{19}$$

Very far from the sphere, the potential is the one created by a point charge of charge:

$$Q = R\varepsilon_0\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi'\, V(\theta',\varphi'), \tag{20}$$

which is the total charge of the sphere. This is in fact true for any charged object of finite size: at very large distance, it behaves as a point charge.

c) Now we have an explicit expression for $V(\theta, \varphi)$:

$$\begin{cases} V(\theta,\varphi) = V_0 + C \text{ for } \cos\theta > 0 \\ V(\theta,\varphi) = C \text{ for } \cos\theta < 0. \end{cases} \tag{21}$$

We need to determine the constant C knowing that the total charge of the sphere is zero. From previous equation (20), we can see that:

$$Q = R\epsilon_0 \int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \, V(\theta', \varphi') = 2\pi R\epsilon_0(2C + V_0) = 0, \tag{22}$$

from which we can deduce $C = -V_0/2$ *i.e.*

$$\begin{cases} V(\theta,\varphi) = \frac{V_0}{2} \text{ for } \cos\theta > 0 \\ V(\theta,\varphi) = -\frac{V_0}{2} \text{ for } \cos\theta < 0. \end{cases} \tag{23}$$

We can use the equation (17) with the explicit function V and we get:

$$\begin{aligned} \Phi(\mathbf{r}) =& \frac{R(r^2-R^2)}{4\pi} \left[\int_0^1 d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \left[\frac{V_0}{2(r^2+R^2-2rR\cos\gamma)^{\frac{3}{2}}} \right] \right. \\ & \left. + \int_{-1}^{0} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \left[-\frac{V_0}{2(r^2+R^2-2rR\cos\gamma)^{\frac{3}{2}}} \right] \right] \end{aligned} \tag{24}$$

It is difficult to simplify this expression in the general case but we can expand in the limit $r \to \infty$. One has:

$$\frac{1}{(r^2+R^2-2rR\cos\gamma)^{\frac{3}{2}}} = \frac{1}{r^3}\frac{1}{(1+x^2-2x\cos\gamma)^{\frac{3}{2}}} = \frac{1}{r^3}(1+3x\cos\gamma) + \mathcal{O}(x^2) \tag{25}$$

where $x = R/r$. In the integral (24), this gives:

$$\begin{aligned} \Phi(\mathbf{r}) =& \frac{R}{4\pi r} \left[\int_0^1 d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \frac{V_0}{2}\left(1 + \frac{3R}{r}\cos\gamma\right) \right. \\ & \left. - \int_{-1}^{0} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \frac{V_0}{2}\left(1 + \frac{3R}{r}\cos\gamma\right) \right] + \mathcal{O}(x^2). \end{aligned} \tag{26}$$

Using the explicit form of $\cos\gamma$, one can calculate that:

$$\int_0^1 d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \cos\gamma = -\int_{-1}^{0} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \cos\gamma = \pi\cos\theta. \tag{27}$$

Putting all together, the leading order terms cancel and we can conclude:

$$\Phi(\mathbf{r}) \to \frac{3V_0R^2}{4r^2}\cos\theta = 3V_0R^2\frac{z}{4r^3}. \tag{28}$$

We find that at very large distance, the sphere behaves like an electrostatic dipole of dipole moment $\mathbf{d} = 3\pi V_0\epsilon_0 R^2 \hat{z}$.

This is in fact a very general statement: a finite size object with zero charge behaves at large distances like a dipole, if it has a non zero dipole moment.

d) We could compute the charge density on the surface of the sphere in the limit $d \to 0$ by taking the formula (24), deriving the electric field at the surface and get the surface density by Gauss theorem but the integrals are difficult to compute explicitly. However, most of the charges are concentrated on the two disks around $\cos\theta = 0$ (the (x, y) plane) and not on the surface of the sphere.

The area around $z = 0$ can be approximated by two planes, separated by a distance d with potential $V_0/2$ and $-V_0/2$. The volume between the two half-balls is subjected to an electric field $\mathbf{E} = -V_0/d\,\hat{z}$ (neglecting boundary effects) so the upper half-ball has a surface charge of $\sigma \approx \epsilon_0 V_0/d$.

The charge of each half-ball is *divergent* when d goes to zero because two regions of different potential are getting very close together. For small d, the charge of the upper half-ball can be approximated by:

$$Q \approx \epsilon_0 \pi R^2 \frac{V_0}{d}\,. \tag{29}$$

One can check by direct computation or simply by dimensional analysis that the absolute value $|Q_C|$ of the total charge residing on the curved part of each hemisphere is of order $|Q_C| \sim \epsilon_0 R V_0$, which is indeed much smaller than the total charge on each disc we found above when $d \to 0$, justifying the claim we made earlier.

e) The problem is the same as the one of point **a)** but we want to find the potential inside the sphere, mathematically in the space $E = \{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r}| \le R\}$ of boundary $\partial E = \{\mathbf{r} \in \mathbb{R}^3 \,:\, |\mathbf{r}| = R\}$.

The first step is to build a Green function G satisfying:

$$\begin{cases} -\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}') & \text{for}\,\mathbf{r} \in E \\ G(\mathbf{r}, \mathbf{r}')\big|_{|\mathbf{r}|=R} = 0, \end{cases} \tag{30}$$

One can use again the image charge method and find that the Green function found in question **a)** solves the problem:

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi}\left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{R}{r'}\frac{1}{|\mathbf{r} - \mathbf{r}''|}\right]. \tag{31}$$

If $\mathbf{r}' \in E$ then $\mathbf{r}'' \notin E$ so this function solves equation (30).

Then using the general formula at the beginning, since the sphere is empty of charges, only one term remains:

$$\Phi(\mathbf{r}) = -\int_{\partial V} [\Phi(\mathbf{r}')\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}')] \cdot \mathbf{d}\sigma'\,. \tag{32}$$

but here

$$\mathbf{d}\sigma' = +R^2 \sin\theta' d\theta' d\varphi' \hat{r}' \tag{33}$$

because the surface vector points outside the volume E. Since $\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}')$ has been computed before, we can conclude that:

$$\Phi(\mathbf{r}) = \frac{R}{4\pi}\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \left[V(\theta', \varphi')\frac{R^2 - r^2}{(r^2 + R^2 - 2rR\cos\gamma)^{\frac{3}{2}}}\right]. \tag{34}$$

At the center of the sphere, $r = 0$ and we have:

$$
\begin{aligned}
\Phi(0) =& \frac{R}{4\pi}\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \left[V(\theta',\varphi') \frac{R^2}{(R^2)^{\frac{3}{2}}} \right] \\
=& \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta' \int_{-\pi}^{\pi} d\varphi' \, V(\theta',\varphi') \, . \qquad (35)
\end{aligned}
$$

The potential at the center of the sphere is the average of the potential on the sphere.

3. Consider the general expressions for the potentials Φ and $\mathbf{A}$ using the retarded Green function,

$$
\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\rho\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} \, , \qquad (36)
$$

$$
\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\mathbf{J}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} \, . \qquad (37)
$$

Show that the Lorenz gauge condition is implied by the continuity equation, *i.e.* charge conservation.

Solution

We start by assuming the continuity equation and show that Φ and $\mathbf{A}$ satisfy the Lorenz gauge condition. The Lorenz gauge condition is

$$
\frac{1}{c^2}\frac{\partial\Phi}{\partial t} = -\nabla\cdot\mathbf{A} \, . \qquad (38)
$$

We start by focusing on the left hand side of (38):

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial\Phi}{\partial t} &= \frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\frac{\partial\rho}{\partial t}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} = \\
&= \frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\frac{\partial\rho}{\partial t'}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} \, . \qquad (39)
\end{aligned}
$$

We were able to simply substitute a derivative with respect to t with a derivative with respect to t' because $\frac{\partial t'}{\partial t} = 1$. Now we use the continuity equation:

$$
\begin{aligned}
&\frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{\frac{\partial\rho}{\partial t'}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} = \\
=&\frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x' \, \frac{-\nabla^{part}_{\mathbf{x}'}\cdot\mathbf{J}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|} \, . \qquad (40)
\end{aligned}
$$

Here the divergence $\nabla^{part}_{\mathbf{x}'}\cdot\mathbf{J}$ means that derivatives of the nabla operator are applied only to the variable $\mathbf{x}'$ and not to the $\mathbf{x}'$ dependence inside t'. In what follows, the symbol $\mathbf{J}$ stands for $\mathbf{J}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)$. Notice that

$$
(\nabla_{\mathbf{x}'} + \nabla_{\mathbf{x}}) f(\mathbf{x}-\mathbf{x}') = 0 \, , \qquad (41)
$$

for any scalar function f of the difference $\mathbf{x}-\mathbf{x}'$. Therefore

$$\frac{\nabla^{part}_{\mathbf{x}'}\cdot\mathbf{J}\left(\mathbf{x}',t'=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}=(\nabla_{\mathbf{x}'}+\nabla_{\mathbf{x}})\cdot\left[\frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}\right]. \tag{42}$$

Indeed, one can develop the r.h.s. as:

$$\begin{aligned}
&(\nabla_{\mathbf{x}'}+\nabla_{\mathbf{x}})\cdot\left[\frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}\right]\\
&\quad=\frac{\nabla^{part}_{\mathbf{x}'}\cdot\mathbf{J}\left(\mathbf{x}',t'=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}\\
&\qquad+\frac{\partial_{t'}\mathbf{J}\left(\mathbf{x}',t'=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}\cdot(\nabla_{\mathbf{x}'}+\nabla_{\mathbf{x}})\left(t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\\
&\qquad+\mathbf{J}\left(\mathbf{x}',t'=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\cdot(\nabla_{\mathbf{x}'}+\nabla_{\mathbf{x}})\frac{1}{|\mathbf{x}-\mathbf{x}'|},
\end{aligned} \tag{43}$$

and the gradients of the last two lines vanish.

We can now go back to the initial integral and proceed with the calculation:

$$\begin{aligned}
&\frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{-\nabla^{part}_{\mathbf{x}'}\cdot\mathbf{J}\left(\mathbf{x}',t'=t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}=\\
&=-\frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x'\nabla_{\mathbf{x}'}\cdot\left[\frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}\right]+\\
&\quad-\frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x'\nabla_{\mathbf{x}}\cdot\left[\frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}\right]=\\
&=-\nabla_{\mathbf{x}}\cdot\frac{1}{c^2}\frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\mathbf{J}}{|\mathbf{x}-\mathbf{x}'|}\right]=-\nabla_{\mathbf{x}}\cdot\mathbf{A}\,.
\end{aligned} \tag{44}$$

In this last derivation, we have used the Gauss' theorem for the integral with the divergence with respect to $\mathbf{x}'$. In this way, we obtain a flux through a surface which is sent to infinity, which is zero if $\mathbf{J}$ goes to zero at large distances.

4. The Faraday disk is a simple electrical generator. The basic ingredients are depicted in 1. There is a metallic annulus with internal radius r_1, external radius r_2 and width h. The annulus is placed in a region with a constant magnetic field perpendicular to the plane of the annulus. The metal of the annulus has electrical conductivity σ.

a) When the annulus is rotating with angular velocity ω and the external circuit is open, what is the electrostatic potential difference V_0 between the internal and the external circumference of the annulus?

b) This potential difference can be used as an electrical generator if we connect the internal and the external circumference of the annulus to a circuit as shown in figure 1 (assume that the connection to the external circuit does not break the cylindrical symmetry of the system). What is the potential

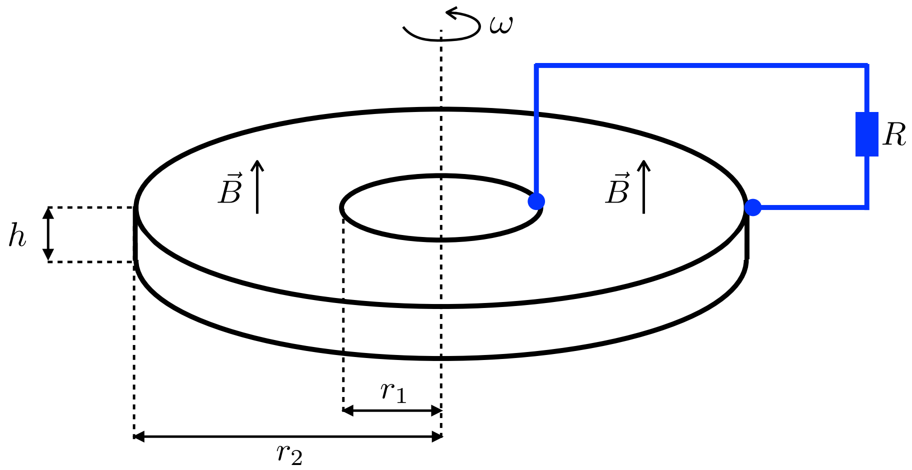

Figure 1: Metallic annulus rotating in a constant magnetic field $\vec{B}$.

difference V (between r_1 and r_2) when there is a current I flowing through the circuit?
Hint 1: Recall the microscopic definition of the conductivity σ,

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) . \tag{45}$$

Hint 2: It might be useful to determine the resistance of the annulus.

c) What is the maximum output of electrical power of this generator? What is the efficiency of the generator when working at the maximal output power?

d) Assume that the metallic annulus is mechanically connected to the wheels of a train so that they rotate with the same angular velocity ω. Assume also that the external electrical circuit acts as an effective resistance R. In this case, we can use the annulus as a regenerative braking system of the train. Start by showing that the total kinetic energy of the train can be written as $E = \frac{1}{2} I_{eff}\omega^2$ and estimate the effective moment of inertia I_{eff}. Then, using conservation of energy, find an equation for the time evolution of the angular velocity ω when the brakes are on, *i.e.* the circuit is closed. How long does it take to reduce the velocity of the train by a factor of 2?

e) **Challenge**: Suppose the annulus is made of $300\,kg$ of iron, the train weighs $400\,ton$ and it is travelling at $200\,km/h$. If we brake the train by short-circuiting the internal and the external circumference of the annulus, will the annulus melt? Is the braking event safe for the passengers?

Solution

a) Consider a point charge of charge q in the metallic annulus. When the circuit is open, there is no current in the annulus so all charges in the conductor are in equilibrium. The charge q feels a force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ which must be zero. Surface charges in the conductor create an electric field of value (in cylindrical coordinates):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} = -r\omega B\hat{r}. \tag{46}$$

From that we get the potential:

$$\Phi = -\int \mathbf{E} \cdot d\mathbf{r} = \frac{r^2}{2}\omega B + const\,, \tag{47}$$

and we can express the potential difference V_0 between r_1 and r_2:

$$V_0 = \Phi(r_2) - \Phi(r_1) = \frac{\omega B}{2}(r_2^2 - r_1^2). \tag{48}$$

b) **Method 1:** Now the circuit is closed and current of intensity I is flowing through the annulus. Assuming cylindrical symmetry, the current density $\mathbf{J}$ can be written as $\mathbf{J} = J(r)\hat{r}$. Consider a cylindrical surface of radius r: what is the current flowing through the surface? It is $2\pi r h J(r) = I$ (here we consider that I is flowing from the internal to the external circumference of the annulus) so:

$$J(r) = \frac{I}{2\pi r h}. \tag{49}$$

Since the problem has a cylindrical symmetry, we can rewrite equation (45) as:

$$\frac{I}{2\pi r h} = \sigma(E(r) + r\omega B), \tag{50}$$

and integrate it from r_1 to r_2. The result is:

$$\frac{I}{2\pi h}\log\left(\frac{r_2}{r_1}\right) = \sigma\left(\Phi(r_1) - \Phi(r_2) + \frac{\omega B}{2}(r_2^2 - r_1^2)\right). \tag{51}$$

From that, we deduce:

$$V = \Phi(r_2) - \Phi(r_1) = \frac{\omega B}{2}(r_2^2 - r_1^2) - \frac{I}{2\pi h\sigma}\log\left(\frac{r_2}{r_1}\right). \tag{52}$$

Method 2: The annulus behaves like a generator but it is not a perfect generator because it has an *internal resistance*. A real generator can be modeled by a perfect generator delivering a tension U_0 and a resistance R_0, the internal resistance. The potential difference between the two ends of the generator is then:

$$V = U_0 - R_0 I. \tag{53}$$

U_0 is the potential difference between the two ends of the generator when zero current is flowing through the circuit: it has been calculated in question a), it is:

$$U_0 = V_0 = \frac{\omega B}{2}(r_2^2 - r_1^2). \tag{54}$$

Now R_0 is the internal resistance of the generator, in this case the resistance of the annulus. It can be computed as follows: remember that a wire made of a metal of conductivity σ has resistance $R = L/\sigma S$ where L is the length of the wire and S the surface of a section. Considering the annulus as many cocentric cylinders of radius r and thickness dr of resistance:

$$dR = \frac{dr}{2\pi r h\sigma}, \tag{55}$$

we get the total resistance by integrating from r_1 to r_2 (we sum resistances in series) and we get:

$$R_0 = \int_{r_1}^{r_2} dR = \frac{1}{2\pi h\sigma} \log\left(\frac{r_2}{r_1}\right). \tag{56}$$

We recognise the result of the first method.

c) The electrical output power of the generator is:

$$P = VI = V_0 I - R_0 I^2 \tag{57}$$

which reaches a maximum for:

$$I = \frac{V_0}{2R_0}, \qquad P_{\max} = \frac{V_0^2}{4R_0}. \tag{58}$$

The total power of the generator (including power loss due to the internal resistance) is $P_{\mathrm{tot}} = V_0 I$ so the efficiency is:

$$\eta = \frac{P_{\max}}{P_{\mathrm{tot}}} = \frac{1}{2}. \tag{59}$$

d) The kinetic energy of the train is $E_c = \frac{1}{2}Mv^2$ where M is the mass of the train (we neglect the rotating parts). But $v = r_0\omega$ where r_0 is the radius of the wheels. So:

$$E_c = \frac{1}{2}Mr_0^2\omega^2 = \frac{1}{2}I_{eff}\omega^2. \tag{60}$$

The closed circuit can be considered as a perfect generator delivering a tension V_0 and two resistances R_0 and R. The intensity flowing through the circuit is $I = \frac{V_0}{R_0+R}$ and the power dissipated by the brakes is $P = V_0 I$. This power comes from the kinetic energy of the train so using conservation of energy, one can write:

$$\frac{dE_c}{dt} = \frac{1}{2}I_{eff}\frac{d\omega^2}{dt} = -P = -V_0 I = -\frac{V_0^2}{R_0+R} = -\frac{\omega^2 B^2}{4(R_0+R)}(r_2^2 - r_1^2)^2. \tag{61}$$

Defining τ as:

$$\tau = \frac{4I_{eff}(R_0+R)}{B^2(r_2^2 - r_1^2)^2}, \tag{62}$$

we have:

$$\omega(t) = \omega_0 \exp\left(-\frac{t}{\tau}\right). \tag{63}$$

The velocity is proportional to the angular velocity so the time needed to reduce the speed of the train by a factor 2 is:

$$t_{1/2} = \tau \log 2 = \frac{4I_{eff}(R_0+R)}{B^2(r_2^2 - r_1^2)^2} \log 2. \tag{64}$$

e) Here, there is no external resistance R so all the power dissipated by the brakes is converted into heat by resistive heating in the annulus. If we neglect dissipation of heat, the thermal energy accumulated in the annulus is the total kinetic energy of the train:

$$\Delta E = \frac{1}{2} M v^2 = 6.2 \times 10^8\,\mathrm{J}. \tag{65}$$

Let us compare this number to the energy needed to heat 300 kg of iron to its melting point at 1800K (the heat capacity of iron is about 0.45 J/g/K):

$$\Delta E = C_{\mathrm{Fe}} m \Delta T \approx 2 \times 10^8\,\mathrm{J}, \tag{66}$$

so the annulus will melt, at least partially. However, we can argue that a train will have one such brake per wheel, say N brakes, and this divides the heat dissipated in each annulus by N. For N of the order of $N = 10$, we are below the melting point.

Another problem is how brutal the braking system is. The acceleration is maximal at the beginning of the braking and is given by:

$$a = \left.\frac{dv}{dt}\right|_{t=0} = -\frac{r_0 \omega_0}{\tau} = -\frac{B^2 (r_2^2 - r_1^2)^2 v_0}{4 I_{eff} R_0} = -\frac{B^2 (r_2^2 - r_1^2)^2 v_0}{4 M r_0^2 R_0}. \tag{67}$$

Now we need to make a certain number of (reasonable) assumptions to estimate this effect. Let us take: $r_0 = 0.5$ m, $r_2 = 0.35$ m, $r_1 = 0.05$ m, $h = 0.1$ m, and B in the range 0.1 to 0.5 T, one can find $\sigma = 1.0 \times 10^7$ S/m. This gives an internal resistance of $R_0 = 3.1 \times 10^{-7}\,\Omega$, a characteristic breaking time of

$$\tau \approx (34 - 860)\,\mathrm{s}\,, \tag{68}$$

and an acceleration of

$$a \approx (0.06 - 1.6)\,\mathrm{m} \cdot \mathrm{s}^{-2}\,. \tag{69}$$

This acceleration seems safe for the passengers (recall that gravity gives $g \approx 10\,\mathrm{m} \cdot \mathrm{s}^{-2}$) but the breaking time is probably too large for an emergency breaking system. Putting N brakes will also solve this problem.

Knowing that the passengers lives are at stake, we must perform a more serious study before building such a train.