Classical Electrodynamics

Solutions week 4

1. Two spherical cavities, of radii a and b, are hollowed out from the interior of a neutral conducting sphere of radius R. At the center of each cavity a point charge is placed. Call these charges q_a and q_b.

a) Find the surface charges σ_a, σ_b and σ_R.

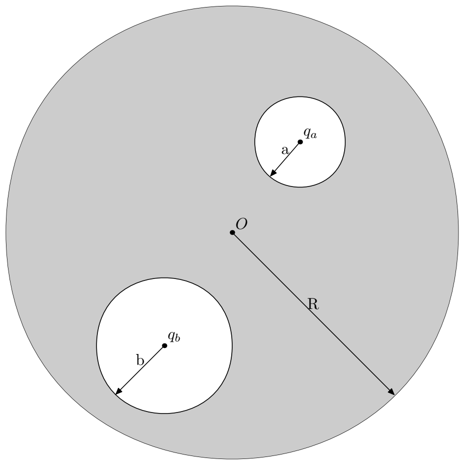
b) What is the field outside the conductor?

c) What is the field within each cavity?

d) What is the force on q_a and q_b?

e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

Hint: it is possible to solve this exercise without any computation beyond elementary algebra.



Solution

a) The electric field inside the conductor is zero. Consider a Gaussian sphere around each cavity. Since the Gaussian surfaces are in the conductor, the total flux of electric field on the spheres vanish. As a result, the total charge inside the sphere is zero which means the total surface charge of each cavity is equal to the charge inside the cavity with a negative sign. Regarding the distribution of the surface charges, one obvious solution is uniform distribution. But due to uniqueness of the solutions, that is the only solution! Therefore, we have:

$$\sigma_a = -\frac{q_a}{4\pi a^2} \tag{1}$$

$$\sigma_b = -\frac{q_b}{4\pi b^2} \tag{2}$$

Since the total charge on the conductor is zero, there must be total charge of $q_a + q_b$ on the surface of conductor at $r = R$. With a similar argument as above, one can conclude the surface charge density is also uniform on the boundary at $r = R$. So:

$$\sigma_R = \frac{q_a + q_b}{4\pi R^2} \tag{3}$$

To justify the uniform distribution, one can also remember that a spherical shell of uniform surface charge produces, outside of the shell, the same electric field as a point of the same total charge located in its center (and produces zero electric field inside the shell). Thus, inside the conductor, the electrici field is indeed zero: the field produced by q_a is compensated by the field of σ_a, same for q_b and σ_b, and σ_R produces zero field.

b) The field is generated by the surface charges at $r = R$, $r = a$ and $r = b$ and also two charges inside the cavities. Because the surface charges at $r = a$ and $r = b$ cancel the effect of the charges q_a and q_b inside cavities, the field outside the conductor is equal to fields generated by surface charges at $r = R$:

$$\mathbf{E}_{out} = \frac{q_a + q_b}{4\pi\epsilon_0 r^2}\hat{r} \tag{4}$$

Note that it does not matter how big the cavities are and where they are located inside the sphere. The conducting sphere is like a Faraday cage.

c) Let's first consider cavity a. The same argument works for cavity b as well. The field generated by the surface charges at $r = R$ and $r = a$ is zero. Moreover the field generated by surface charges at $r = b$ cancels the contribution from the charge q_b. So the field inside the cavity i is:

$$\mathbf{E}_i = \frac{q_i}{4\pi\epsilon_0 r^2}\hat{r} \tag{5}$$

where the origin of the spherical coordinates is the charge q_i.

d) As discussed above, the electric field generated by the surface charges and $q_{b(a)}$ inside the cavity $a(b)$ vanishes. As a result, the force on q_a and q_b is zero. Note that this would not be true if the charges were not at the center of the spherical cavities. Say q_a is not at the center of cavity a, then the suface charge σ_a would be stronger at positions closer to q_a, and cause an attraction of the charge q_a towards the wall of the cavity.

e) Somehow σ_R and the field outside the conductor would change to cancel the effect of q_c inside the condutor, so that the condition $\mathbf{E}_{\text{conductor}} = 0$ is satisfied. On the other hand, σ_a, σ_b, the electric field within each cavity and the absence of forces on the charges q_a and q_b will not change.

Note that one can also use the uniqueness of the solution to answer all of the above questions.

2. Given the charge density $\rho(x, y, z)$ and the value of the potential $\Phi(x, y, 0) = \varphi(x, y)$ on the plane $z = 0$, determine the potential $\Phi(x, y, z)$ in the region $z \geq 0$.

Hint: Start by finding the appropriate Green function $G(\mathbf{r}, \mathbf{r}')$ for this problem. Recall the general solution of Poisson equation

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}', \mathbf{r}) d^3\mathbf{r}' + \int_{\partial V} [G(\mathbf{r}', \mathbf{r}) \nabla_{\mathbf{r}'} \Phi(\mathbf{r}') - \Phi(\mathbf{r}') \nabla_{\mathbf{r}'} G(\mathbf{r}', \mathbf{r})] \cdot \mathbf{d}\sigma' \,. \tag{6}$$

Solution

We want to solve the Poisson equation $\nabla^2 \Phi = \rho/\epsilon_0$ in the upper half space $V = \{\mathbf{r} = (x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ with boundary given by the plane $\partial V = \{\mathbf{r} \in \mathbb{R}^3 : z = 0\}$.

We start by reminding that the solution to the Poisson equation is given by equation (6) where the Green function $G(\mathbf{r}, \mathbf{r}')$ is solution of the equation

$$\begin{cases} \nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = -\delta^3(\mathbf{r} - \mathbf{r}') \\ G(\mathbf{r}, \mathbf{r}')\big|_{\mathbf{r} \in \partial V} = 0 \,, \end{cases} \tag{7}$$

where the second condition is added because it is the most convenient for Dirichlet boundary conditions. Notice that the term involving the normal derivative of Φ in (6) drops out with this choice. We are going to guess such a Green function using the method of image charge.

The first equation can be interpreted as the potential generated in $\mathbf{r}$ by a point charge of unit value placed at position $\mathbf{r}'$. Moreover, we want the potential satisfying the boundary condition that it vanishes at the surface $z = 0$.

We already know how to solve this problem with the image charge method. Here, $G(\mathbf{r}, \mathbf{r}')$ is given by the potential generated by a charge at position $\mathbf{r}' = (x', y', z')$ and a negative unit charge at $\mathbf{r}'' = (x', y', -z')$, symmetric of $\mathbf{r}$ with respect to the $z = 0$ plane. Therefore, the Green function we are looking for is

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') = & \frac{1}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \\ & - \frac{1}{4\pi\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}} \,. \end{aligned} \tag{8}$$

It is easy to show that $G(\mathbf{r}, \mathbf{r}') = 0$ for $z = 0$. The student may also check that, in the region $z \geq 0$, we have $\nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$. In addition, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$ as it must for a Dirichlet Green function.

Now, if $\hat{\mathbf{n}}$ is a unitary vector perpendicular to the plane and pointing toward the negative values of the z-axis, we can write

$$\nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{d}\sigma' = \nabla_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') \cdot \hat{\mathbf{n}} \, dx' dy' = -\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial z'} \, dx' dy' \,. \tag{9}$$

Since the derivative of the Green function with respect to z' in $z' = 0$ is

$$\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial z'}\bigg|_{z'=0} = \frac{2z}{4\pi \left[(x - x')^2 + (y - y')^2 + z^2\right]^{3/2}} \,, \tag{10}$$

the result is

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}', \mathbf{r}) d^3\mathbf{r}' + \\ + \frac{2z}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\varphi(x', y')}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} dx' dy' \,. \tag{11}$$

3. Consider the region $\rho \leq \rho_0$ and $0 \leq \varphi \leq \beta$ in cylindrical coordinates as depicted in figure 1. There are no charges inside this region and the potential is fixed at the boundaries to $\Phi(\rho, \varphi = 0, z) = \Phi(\rho, \varphi = \beta, z) = 0$ and $\Phi(\rho_0, \varphi, z) = V(\varphi)$, where $V(\varphi)$ is a continuous function for $0 \leq \varphi \leq \beta$ and $V(0) = V(\beta) = 0$.

a) Check that the potential inside the region is given by

$$\Phi(\rho, \varphi, z) = \sum_{m=1}^{\infty} a_m \left(\frac{\rho}{\rho_0}\right)^{m\pi/\beta} \sin \frac{m\pi\varphi}{\beta} \tag{12}$$

and determine the coefficients a_m in terms of V.

b) Determine the surface charge density $\sigma(\rho)$ along the boundary $\varphi = 0$ of the grounded conductor.

c) Assuming the generic case $a_1 \neq 0$, study the behaviour of the surface charge density $\sigma(\rho)$ near the corner at $\rho = 0$. Comment on the 4 cases $0 < \beta < \pi$, $\beta = \pi$, $\pi < \beta < 2\pi$ and $\beta = 2\pi$ and their physical meaning.

d) **Challenge**: Take $\beta = 3\pi/2$ and suppose the region is filled with air. What will happen near the corner at $\rho = 0$?

Solution

a) In order to check that the solution for the potential is correct, we need to do two things:
i) Show that the boundary conditions are satisfied.
ii) Show that the Poisson equation is satisfied.

i) First of all, we see that $\Phi(\rho, 0, z) = \Phi(\rho, \beta, z) = 0$. Secondly, we need to find the coefficients a_m so that

$$\Phi(\rho_0, \varphi, z) = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi\varphi}{\beta} = V(\varphi) \,. \tag{13}$$

We can do this by multiplying equation (13) on both sides for $\sin \frac{l\pi\varphi}{\beta}$, where

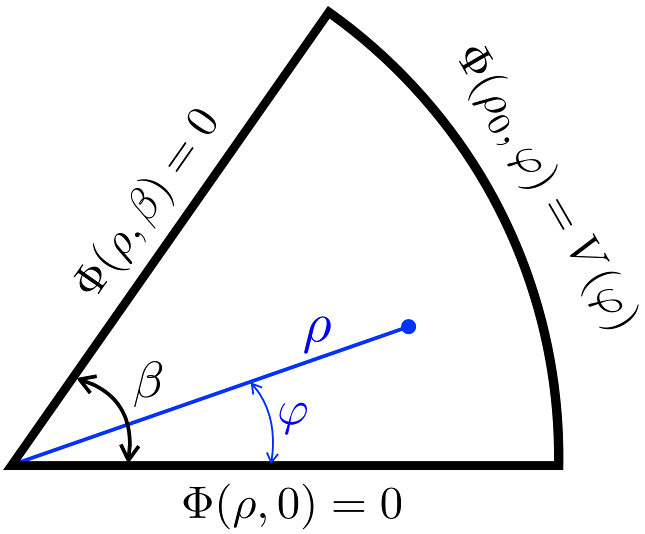


Figure 1: Two dimensional cross section of a region that extends along the z-direction orthogonal to the plane of the figure. The system is invariant under translations in the z-direction. The radial conductors at $\varphi = 0$ and $\varphi = \beta$ are grounded. The surface at $\rho = \rho_0$ has a given potential profile $V(\varphi)$. The angular region bounded by these 3 surfaces is empty (vacuum).

l is a positive integer, and integrating between 0 and β:

$$\begin{aligned}
&\int_0^\beta d\varphi \sum_{m=1}^{\infty} a_m \sin\frac{m\pi\varphi}{\beta} \sin\frac{l\pi\varphi}{\beta} = \\
&= \sum_{m=1}^{\infty} a_m \int_0^\beta d\varphi \frac{1}{2}\left[\cos\frac{(m-l)\pi\varphi}{\beta} - \cos\frac{(m+l)\pi\varphi}{\beta}\right] = \\
&= \sum_{m=1}^{\infty} \frac{a_m}{2} \int_0^\beta d\varphi \left[\cos\frac{(m-l)\pi\varphi}{\beta}\right] = \\
&= \sum_{m=1}^{\infty} \frac{a_m}{2}\beta\delta_{ml} = a_l\frac{\beta}{2} = \\
&= \int_0^\beta d\varphi\, V(\varphi) \sin\frac{l\pi\varphi}{\beta} \;.
\end{aligned} \tag{14}$$

From (14), we deduce the coefficients:

$$a_l = \frac{2}{\beta}\int_0^\beta d\varphi\, V(\varphi)\sin\frac{l\pi\varphi}{\beta} \;. \tag{15}$$

ii) Now we need to prove that

$$\nabla^2\Phi(\rho,\varphi,z) = 0 \;. \tag{16}$$

Using cylindrical coordinates, the Laplacian becomes

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Phi}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} \;. \tag{17}$$

The first term on the right hand side of (17) is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Phi}{\partial\rho}\right] = \sum_{m=1}^{\infty} a_m \left[\frac{m\pi}{\beta}\right]^2 \frac{1}{\rho_0^2}\left(\frac{\rho}{\rho_0}\right)^{m\pi/\beta-2} \sin\frac{m\pi\varphi}{\beta} \;. \tag{18}$$

When we evaluate the derivatives in the second term, we obtain

$$\begin{aligned}\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} &= -\sum_{m=1}^{\infty} a_m \left[\frac{m\pi}{\beta}\right]^2 \frac{1}{\rho_0^2}\left(\frac{\rho}{\rho_0}\right)^{m\pi/\beta-2} \sin\frac{m\pi\varphi}{\beta} = \\ &= -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\Phi}{\partial\rho}\right] \;.\end{aligned} \tag{19}$$

Plugging (18) and (19) into (17) we obtain (16).

b) Let us first compute the electric field in the region :

$$\mathbf{E}(\rho,\varphi,z) = -\nabla\Phi = -\sum_{m=1}^{\infty} a_m \frac{m\pi}{\beta\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{m\pi}{\beta}-1}\left[\sin\frac{m\pi\varphi}{\beta}\hat{\mathbf{e}}_\rho + \cos\frac{m\pi\varphi}{\beta}\hat{\mathbf{e}}_\varphi\right],$$

which, along the boundary $\varphi = 0$ reduces to:

$$\mathbf{E}(\rho) = -\sum_{m=1}^{\infty} a_m \frac{m\pi}{\beta\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{m\pi}{\beta}-1}\hat{\mathbf{e}}_\varphi, \tag{20}$$

and thus:

$$\sigma(\rho) = -\epsilon_0 \sum_{m=1}^{\infty} a_m \frac{m\pi}{\beta\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{m\pi}{\beta}-1} \;. \tag{21}$$

c) Assuming that $a_1 \neq 0$, we have for $\rho \to 0$:

$$\sigma(\rho) = -\epsilon_0 a_1 \frac{\pi}{\beta\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{\pi}{\beta}-1} + \mathcal{O}\left(\frac{\rho}{\rho_0}\right)^{\frac{2\pi}{\beta}-1} \;. \tag{22}$$

The behaviour of $\sigma(\rho)$ for $\rho \to 0$ can be deduced from the leading term. In each case:

- $0 < \beta < \pi$: in this case, the conductor has a inwards corner. The leading exponent $\frac{\pi}{\beta} - 1$ is positive and $\sigma(\rho)$ goes smoothly to zero, there is no special behaviour at the corner.
- $\beta = \pi$: the conductor is a semi-circle and there is no discontinuity at the point $\rho = 0$. The leading exponent is zero $\frac{\pi}{\beta} - 1 = 0$ and the charge density at the center is $\sigma(0) = -\frac{\epsilon_0 a_1}{\rho_0}$. Moreover, all physical quantities are analytic in ρ.
- $\pi < \beta < 2\pi$: in this case, the conductor has a corner at $\rho = 0$. The leading exponent $\frac{\pi}{\beta} - 1$ is negative and the charge density diverges at the corner.
- $\beta = 2\pi$: the conductor is reduced to a needle. The leading exponent is $\frac{\pi}{\beta} - 1 = -\frac{1}{2}$ and the charge density diverges at the tip of the needle like $\frac{1}{\sqrt{\rho}}$.

Notice that the total charge (per unit length along the z-direction)

$$\int_0^{\rho_0} d\rho\, \sigma(\rho) = -\epsilon_0 \sum_{m=1}^{\infty} a_m = -\epsilon_0 \frac{1}{\beta} \int_0^{\beta} d\varphi V(\varphi) \cot \frac{\pi\varphi}{2\beta} \tag{23}$$

is finite for any β, even though the charge density diverges for $\beta > \pi$.

d) $\beta = \frac{3\pi}{2}$ is in the third case: the charge density and also the electric field are divergent at the corner. In practice, when the electric field reaches a critical value, it can ionize the air around the corner. The ions then move between the corner of the conductor and the $\rho = \rho_0$ boundary (from high potential to low potential for positive ions). Since the potentials are held fixed, this creates a steady current. The ionization around the corner is accompanied by scattering of charged particles on nuclei, which creates a purple glow and buzzing noise. This effect is called **corona discharge** or **corona effect**.

A quantitative treatment of this effect is beyond the scope of this course.