Classical Electrodynamics

Week 3

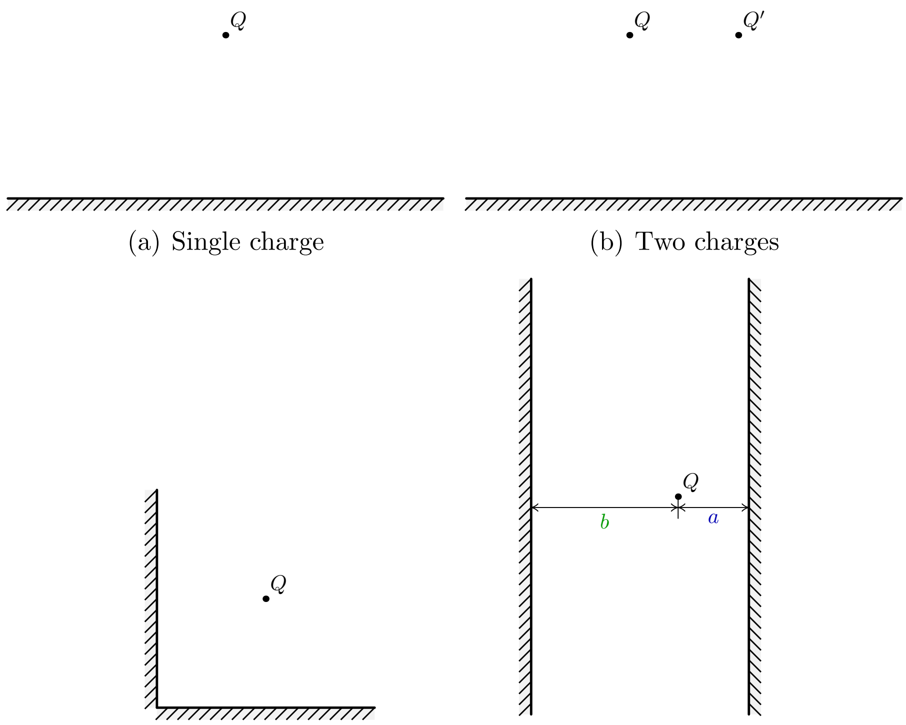
1. In order to get familiar with the method of image charges it's better to warm up with some examples. For all the following configurations draw the position and the value of the image charge and check that the scalar potential ϕ is zero on the surface of the conductors.

a) A single point–like charge and an infinite conducting plane (figure 1(a)).

b) Two point–like charges of different values and an infinite conducting plane (figure 1(b)).

c) A single charge placed inside a 90 degrees corner of conducting materials (figure 1(c)).

d) A single charge between two parallel metallic planes (figure 1(d)). For this last one it is not required to compute the full solution but only to discuss qualitatively the configuration of image charges that realizes it.



(a) Single charge (b) Two charges

(c) A corner (d) Parallel planes

Figure 1: Configurations of charges and conducting planes.

Solution

a) We need to replace the conducting plane with an appropriately positioned image charge such that the electric field or the potential in the area of interest is the same. To do so, it is sufficient to check that the boundary potential is the same and that the charge distribution in the space of interest is not altered.

In this case, as the potential on a conductor is constant (otherwise charges would be flowing), and as the conductor extends to infinity where the potential is zero (by convention), one needs to place the image charge such that the potential on a virtual plane (the boundary of the conductor) is zero.

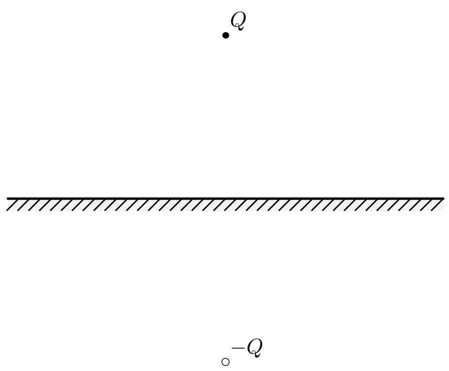
The solution is shown in figure 2(a). All the points on the plane are equidistant to the two opposite charges. As the potential is the sum of the potential created by each of the charges, the potential on the plane amounts to zero.

b) In this case (2(b)), the potential outside the conductor is the sum of the potential of each of the 4 charges. On the plane, the potential created by the upper-left charge cancels with the one created by the lower-left image-charge and the potential of the upper-right with the one of the lower-right.

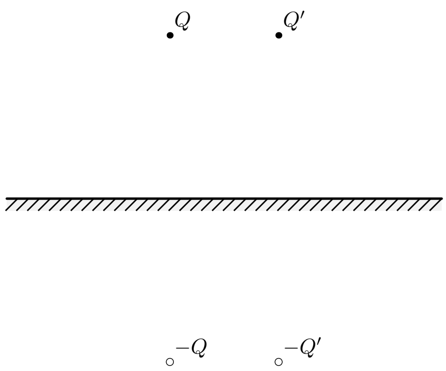
c) On the vertical plane, the potential created by the upper-left charge and the upper-right charge cancel, so does the potential created by the lower-left and lower-right charges. On the horizontal plane, the potential created by the upper-right and the lower-right charges cancel, and the potential created by the upper-left and lower-left charges cancel.

d) In this configuration (2(d)), the potential on right-hand plane is zero (the left part is a mirror image of the right with opposite charges). But the potential on the left plane is not zero (it is the sum of the two charges on the right). To make it zero, we should add two charges on the left so we have again mirror images (in respect to a "mirror" on the left plane). In that case, the potential on the right plane would not be zero but would be the potential created by the charges we have just added. One sees that we could repeat this process at infinity where the "unwanted" potential would be the potential created by two lastly-added charges but so far away that this potential would tend to zero!

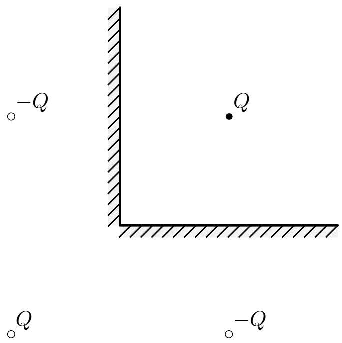
If we place the original charge at the origin and put the two conducting planes at $x = a$ and $x = -b$, we need an infinite number of image charges Q placed at $x = 2(a+b)k, y = 0, z = 0$ where $k \in \mathbb{Z}$ ($k = 0$ is the original charge, all other are image charges) and another infinite number of image charges $-Q$ at coordinates $x = 2a + 2(a+b)k,\ k \in \mathbb{Z}$. One might worry about the convergence of the solution found by the image charge method. This is left as an exercise to the reader.



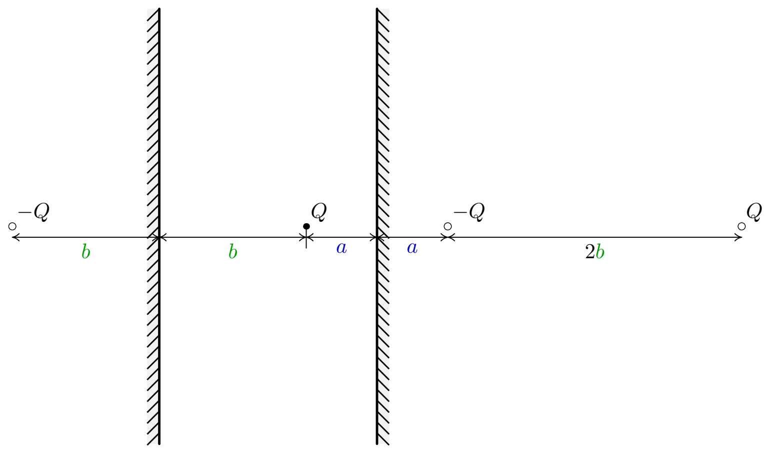
(a) Single charge



(b) Two charges



(c) A corner



(d) Parallel planes

Figure 2: Configurations of charges and conducting planes, along with the image charges. The potential created by the charges and their images (without the conductors) is equivalent to the potential created by the charge and the conducting planes.

2. Consider a conducting sphere of radius R and a point-like charge Q placed at a distace d from the surface of the sphere.

a) Find the position and the value of the image charge inside the sphere such that the potential ϕ is zero on the surface of the sphere.

b) Find how to place an additionnal image charge such that the sphere is neutral and has constant potential on its surface (not necessarily zero).

Solution

a) Let's consider that the centre of the sphere is at the origin and the charge Q is placed at $\mathbf{x_0} = (0, 0, R+d)$. The sphere is a conductor so its surface must be *equipotential*. The problem we have to solve in order to find the potential is : $\nabla^2\phi(\mathbf{x}) = -\frac{Q}{\epsilon_0}\delta(\mathbf{x} - \mathbf{x_0})$ with boundary conditions : ϕ constant on the surface of the sphere. We will use the method of image charges in order to reproduce these conditions.

We assume it is possible to place a single charge inside the sphere so that the potential is zero on the surface. It will make the equations easier to solve[1]. By the cylindrical symmetry of the problem, the mirror charge Q' must be on the z axis, at coordinates $(0, 0, a)$. If we use spherical coordinates (r, θ, φ), we get a potential:

$$\phi(r, \theta, \varphi) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + (R+d)^2 - 2r(R+d)\cos\theta}} + \frac{Q'}{4\pi\varepsilon_0} \frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} \tag{1}$$

Considering first the two points $P_1 = (R, 0, 0)$ and $P_2 = (R, \pi, 0)$, we get a system of two equations with solutions:

$$Q' = -Q\frac{R}{R+d}, \qquad a = \frac{R^2}{R+d} \tag{2}$$

It is then straighforward to check that with these solutions, $\phi = 0$ for $r = R$ for any θ.

b) With the first question we have found a way to get constant zero potential on the surface of the sphere. Now, if we add one more charge Q'' such that the potential from that charge only is constant on the sphere, by superposition principle the potential from all three charges is still a constant. Clearly, the place such a charge can be is the centre of the sphere. We can now adjust the value of Q'' to have a neutral sphere. Intuitively $Q'' = -Q'$. And indeed, Gauss' law (for a sphere of radius $R+\epsilon$) implies that the flux of the electric field outside the sphere indicates the total charge of the sphere is $Q' + Q''$:

$$\Phi(\mathbf{E}) = \frac{1}{\varepsilon_0}(Q' + Q'') \tag{3}$$

[1] If this did not work, we could have tried to equate the potential to an arbitrary constant, which would be a third unknown, so we would need to compare three points. Computations become involved doing so. Alternatively, one can impose the derivatives of ϕ with respect to θ to vanish. This is easiest by looking at the second derivative in θ, at P_1 and P_2.

So we immediately get :

$$Q'' = -Q' = Q\frac{R}{R+d} \tag{4}$$

Conclusion: the potential *outside the conductive sphere* satisfies the same equation with boundary conditions in the original problem and in the simplified case where we only have the three charges Q, Q' and Q''. The potential outside the sphere is thus (valid for $r \geq R$):

$$\phi = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+(R+d)^2-2r(R+d)\cos\theta}} - \frac{R}{R+d}\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} + \frac{R}{R+d}\frac{1}{r}\right] \tag{5}$$

3. Consider a point-like charge Q hanging from an insulating string as shown in figure 3(a).

a) Using Maxwell equations, compute the electric field produced by the charge,

(i) neglecting the effect of all nearby objects.

(ii) assuming that the ground under the charge is a flat infinite conductor at a distance h. Determine also the surface charge density induced on the conducting floor. **Hint**: use the method of image charges.

b) Suppose that the floor is an insulator with the electrical permittivity of the vacuum and we place a neutral metallic sphere on the ground exactly below the hanging charge as depicted in figure 3(b). The sphere has radius R and the distance from the top of the sphere to the hanging charge is d.

(i) Using the method of image charges, show that the surface charge density induced on the conducting sphere is

$$\sigma(\theta) = \frac{Q}{4\pi}\left[\frac{1}{R(d+R)} - \frac{d(d+2R)}{R\left(d^2+4R(d+R)\sin^2\frac{\theta}{2}\right)^{3/2}}\right], \tag{6}$$

where θ is the polar angle from the top of the sphere. **Hint**: use the result of exercise **2.b**

(ii) Study the limit of the charge density for a very large sphere ($R \to \infty$) keeping the distance d fixed. Compare your result with the case of a flat conducting floor.

(iii) What is the minimal weight of the sphere so that it remains on the ground?

c) **Challenge**: Assume the ground in figure 3(a) is a water pond. What is the shape of the surface of the water due to the presence of a small hanging charge?

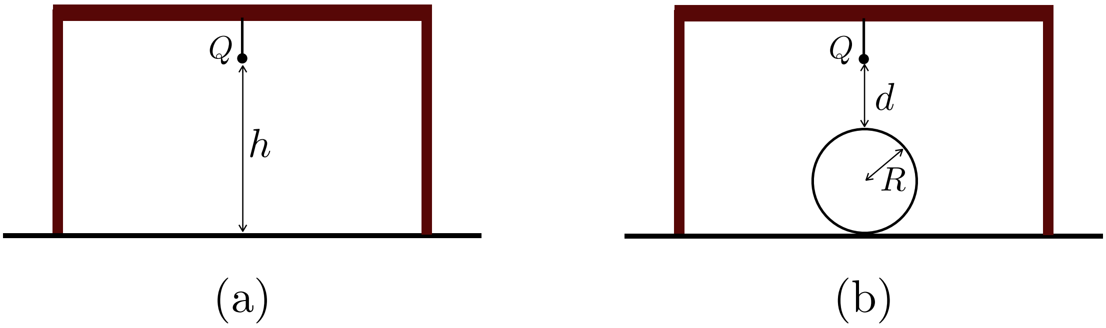


Figure 3: Charge hanging from an insulating string.

Solution

a) (i) Consider a sphere of radius r centered at the position of the particle. According to Gauss's law, we have

$$\Phi(\mathbf{E}) = \frac{Q}{\epsilon_0} \; . \tag{7}$$

Due to the spherical symmetry of the problem, the electric field is always perpendicular to the surface of the sphere: $\mathbf{E} = E\hat{e}_r$. Therefore, the flux is given by

$$\Phi(\mathbf{E}) = \int dS\,\mathbf{E}\cdot\mathbf{n} = \int dS\,E = E4\pi r^2 \; . \tag{8}$$

And thus the electric field is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{e}_r \; , \tag{9}$$

as we should expect from the Coulomb law.

(ii) We choose a reference frame so that the conductor is placed in the plane $z = 0$. Since all the material in $z < 0$ is a conductor, we know that $\mathbf{E} = 0$ for $z < 0$ and thus the potential ϕ is constant for z negative. Since the conductor extends to infinity, we have $\phi = 0$ for $z < 0$.

In order to find the potential for z positive, we have to solve the following problem (Poisson equation with boundary conditions):

$$\begin{cases} \nabla^2\phi = \frac{\rho}{\epsilon_0} \\ \phi(z=0) = 0 \end{cases} \tag{10}$$

We will solve this problem with the method of image charges. The idea is to introduce ficticious charges *outside the volume where we want to solve the Poisson equation* in order to reproduce the boundary conditions (here $\phi = 0$ when $z = 0$).
To do so, we introduce an image charge $q' = -Q$ at distance h on the opposite side of the conductor as on figure 2(a). This does not spoil the Poisson equation for $z > 0$ and on the boundary $z = 0$ we have a potential :

$$\phi(x, y, z = 0) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+h^2}} - \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+h^2}} = 0 \; . \tag{11}$$

This reproduces the boundary conditions and we can conclude that the potential in the whole upper space $z > 0$ is given by the sum of the potentials generated by the two charges:

$$\phi(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z-h)^2}} - \frac{Q}{4\pi\epsilon_0\sqrt{x^2+y^2+(z+h)^2}} \ . \tag{12}$$

The three component of the electric field are derived using $\mathbf{E} = -\nabla\phi$ (or summing the field produced by the two charges):

$$\begin{aligned}
E_x &= \frac{x\,Q}{4\pi\epsilon_0\left(x^2+y^2+(z-h)^2\right)^{3/2}} - \frac{x\,Q}{4\pi\epsilon_0\left(x^2+y^2+(z+h)^2\right)^{3/2}} \ , \\
E_y &= \frac{y\,Q}{4\pi\epsilon_0\left(x^2+y^2+(z-h)^2\right)^{3/2}} - \frac{y\,Q}{4\pi\epsilon_0\left(x^2+y^2+(z+h)^2\right)^{3/2}} \ , \\
E_z &= \frac{(z-h)\,Q}{4\pi\epsilon_0\left(x^2+y^2+(z-h)^2\right)^{3/2}} - \frac{(z+h)\,Q}{4\pi\epsilon_0\left(x^2+y^2+(z+h)^2\right)^{3/2}} \ .
\end{aligned}$$

We will now relate the surface charge with the electric field found above. Notice that the electric field is discontinuous at the surface $z = 0 : E_z(x, y, 0_-) = 0$ and $E_z(x, y, 0_+) \neq 0$.
Consider now a closed box delimited by two large parallel planes placed at $z = \epsilon$ and $z = -\epsilon$, where ϵ is a small positive parameter. We will apply Gauss's law, so we first compute the flux through the box:

$$\begin{aligned}
\Phi(\mathbf{E}) &= \oint dS_{\text{box}}\,\mathbf{E}(\mathbf{x})\cdot\mathbf{n} = \int dS\,E_z(x,y,\epsilon) + \text{flux through the sides} \\
&\underset{\epsilon\to 0}{\to} \int dS\,E_z(x,y,0_+)
\end{aligned} \tag{13}$$

Gauss's law tells us :

$$\Phi(\mathbf{E}) = \frac{Q_{\text{box}}}{\epsilon_0} = \frac{1}{\epsilon_0}\int dS\,\sigma(x,y) = \int dS\,E_z(x,y,0_+) \ . \tag{14}$$

Equation (14) is satisfied for any such surface only if

$$\frac{\sigma(x,y)}{\epsilon_0} = E_z(x,y,0_+) \ . \tag{15}$$

Therefore, the surface density is given by

$$\sigma(x,y) = -\frac{Q\,h}{2\pi\left(x^2+y^2+h^2\right)^{3/2}} \ . \tag{16}$$

Notice that the total charge of the conducting plane is:

$$Q_p = \iint \sigma(x,y)dxdy = -Q \tag{17}$$

exactly the opposite of the hanging charge!

Note : The result we have just derived is more general than this exercise. Using a similar argument, you can show that the surface charge density of any surface is proportionnal to the discontinuity of the electric field normal to the surface:

$$\sigma(\mathbf{x}) = \epsilon_0 \Delta E_\perp(\mathbf{x}) \tag{18}$$

You can use this result later in the course without rederiving it.

b) (i) Let's consider that the centre of the sphere is at the origin and the charge Q is placed at $\mathbf{x_0} = (0, 0, R + d)$. The sphere is neutral and a conductor, so its surface must be *equipotential*. The problem of solving Poisson's equation outside that sphere has been solved with image charges in problem **2.** It involved placing an image charge $Q' = -Q\frac{R}{R+d}$ at position $(0, 0, \frac{R^2}{R+d})$ and one charge $Q'' = -Q'$ at the origin. The potential outside the sphere is given in (5).

Now we can compute the electric field outside the sphere at the surface by taking the gradient, and we deduce the surface charge with the formula $\sigma = \varepsilon_0 \mathbf{E}\cdot\hat{\mathbf{n}}$. We need only the radial component of the gradient:

$$\sigma = \varepsilon_0 \mathbf{E} \cdot \hat{\mathbf{e}}_r = -\varepsilon_0 \nabla\phi \cdot \mathbf{e}_r = -\varepsilon_0 \frac{\partial \phi}{\partial r}, \tag{19}$$

$$\sigma(\theta, \varphi) = -\frac{Q}{4\pi R}\left[\frac{d(d+2R)}{(R^2 + (R+d)^2 - 2R(R+d)\cos\theta)^{\frac{3}{2}}} - \frac{1}{R+d}\right] \tag{20}$$

(ii) Now let $R \to \infty$, d fixed in the previous equation. We get:

$$\sigma(\theta, \varphi) = \frac{Q}{4\pi R^2} \tag{21}$$

so for a very large sphere, we have a constant, very small charge density all over the sphere. But now the total charge of the sphere is Q! What happened?

Notice that when $\theta \to 0$, the denominator of the first fraction cancels. So we have to expand the cosine to investigate the small angle limit : $\cos\theta = 1 - \frac{\theta^2}{2}$. The charge density becomes:

$$\sigma = -\frac{Q}{4\pi}\left[\frac{2d}{(d^2 + R^2\theta^2)^{\frac{3}{2}}}\right] \tag{22}$$

When the sphere becomes very large, most of the sphere gets a small surface charge density of the same sign as the hanging charge except for the tip of the sphere where there is a concentration of opposite signe charges.

Compared to the plane, we can say $R^2\theta^2 \to x^2 + y^2$ when R becomes large so we find the charge distribution for the infinite plane:

$$\sigma = -\frac{Q}{2\pi}\frac{d}{(d^2 + x^2 + y^2)^{\frac{3}{2}}} \tag{23}$$

(iii) The force exerted by the point charge on the sphere is the opposite of the force exerted by the sphere on the point charge. This is given by the force excerced by the electric field created by the sphere on Q. And the electric field produced by the sphere is the same as the one produced by the two image charges Q' and Q''. So the force acting on the sphere is:

$$\mathbf{F} = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{Q'}{(R+d-a)^2} + \frac{Q''}{(R+d)^2}\right]\hat{e}_z = \\ = \frac{Q^2}{4\pi\varepsilon_0}\frac{R^3}{d^2(R+d)}\frac{2d^2+4dR+R^2}{(d+R)^2(d+2R)^2}\hat{e}_z \; . \tag{24}$$

In order for the weight of the sphere to cancel this force, one needs:

$$m > \frac{Q^2}{4\pi g\varepsilon_0}\frac{R^3}{d^2(R+d)}\frac{2d^2+4dR+R^2}{(d+R)^2(d+2R)^2} \; . \tag{25}$$

c) The water can be considered as the infinite conductor of part a). The charge Q will attract the surface charges and the surface of the water will rise. The complete problem is impossible to treat analytically but it is reasonable to assume that the deformation of the water is small. From the electrostatic point of view, the surface of the water remains flat so that the surface charge density is simply:

$$\sigma = -\frac{Q}{2\pi}\frac{d}{(d^2+x^2+y^2)^{\frac{3}{2}}} \; . \tag{26}$$

Each point at the surface of the water is subjected to an electric field: $\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{e}_z$ for $z > 0$ and $\mathbf{E} = 0$ for $z < 0$. Therefore, an element of surface dS feels a force:

$$d\mathbf{F} = \frac{\sigma^2}{2\varepsilon_0}dS\hat{e}_z = \frac{Q^2\,dS}{8\pi^2\varepsilon_0}\frac{d^2}{(d^2+x^2+y^2)^3}\hat{e}_z \; . \tag{27}$$

(since the electric field is discontinuous, one has to consider the average of the value above and below the plane).

This force is cancelled by the weight of the column of water above the original height, given by

$$d\mathbf{P} = -\rho g\delta z(x,y)\,dS\hat{e}_z \; , \tag{28}$$

where ρ is the density of water and $\delta z(x,y)$ is the height of the water with respect to $z = 0$. When we impose $d\mathbf{F} + d\mathbf{P} = 0$, we obtain:

$$\delta z(x,y) = \frac{Q^2}{8\pi^2\varepsilon_0 g\rho}\frac{d^2}{(d^2+x^2+y^2)^3} \; . \tag{29}$$

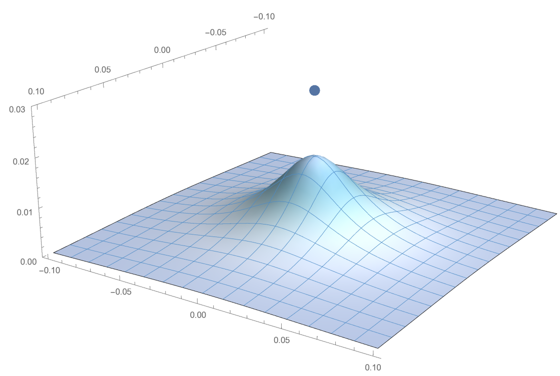


Figure 4: Plot of the shape of the water (z axis exaggerated).