Classical Electrodynamics

Week 2

1. Show that

a) $\delta_{ii} = 3$

b) $\delta_{ij}\epsilon_{ijk} = 0$

c) $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

d) $\epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij}$

Solution

Remember that we use Einstein notation and sum over repeted indices. Moreover, i, j, k are used for euclidean space indices. We furthermore remind you that

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1}$$

and that

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if} \quad (i,j,k) \text{ is} \quad (1,2,3),(2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if} \quad (i,j,k) \text{ is} \quad (1,3,2),(2,1,3) \text{ or } (3,2,1) \\ 0 & \quad \text{otherwise} \end{cases} \tag{2}$$

ie. it measures the sign of the permutation.

a) Explicitly $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

b) We can proceed explicitly $\delta_{ij}\epsilon_{ijk} = \sum_{i=1}^{3} \delta_{ii}\epsilon_{iik}$ because of (1). And then $\sum_{i=1}^{3} \delta_{ii}\epsilon_{iik} = 0$ using (2).

Note that this is true more generally if we contract a symmetric with an antisymmetric tensor. Here $\delta_{ij} = \delta_{ji}$ is symmetric in i, j and $\epsilon_{ijk} = -\epsilon_{jik}$ is antysimetric. Remembering that for a symmetric tensor $S_{ij} = S_{ji}$ whereas for antisymmetric tensor $A_{ij} = -A_{ji}$, it follows that

$$S_{ij}A_{ij} = -S_{ij}A_{ji} = -S_{ji}A_{ji} = -S_{ij}A_{ij} \implies S_{ij}A_{ij} = 0, \tag{3}$$

where in the last step of the equalities we relabelled the indices $i \leftrightarrow j$.

c) One can write all terms explicitly, or we can also understand it as follows. From (2), we have a non-zero situation only in the following two configuration

- $j = m$, $k = n$ and $i \neq j, k$
- $j = n$, $k = m$ and $i \neq j, k$

In the first situation the two epsilon have the same ordering whereas in the second they differ. Thus

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{4}$$

d) We can use the previous identity

$$\begin{aligned}\epsilon_{imn}\epsilon_{jmn} &= \epsilon_{mni}\epsilon_{mnj} = \delta_{kn}(\epsilon_{mni}\epsilon_{mkj}) \\ &= \delta_{kn}(\delta_{nk}\delta_{ij} - \delta_{nj}\delta_{ik}) = \delta_{kn}\delta_{nk}\delta_{ij} - \delta_{kn}\delta_{nj}\delta_{ik} \\ &= 3\delta_{ij} - \delta_{ij} = 2\delta_{ij} \qquad (5)\end{aligned}$$

2. Vector idenities

By looking at a component and without writing the vectors explicitly, prove the following identities:

a) $\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

b) $\nabla \cdot (\mathbf{A} \wedge \mathbf{B}) = (\nabla \wedge \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \wedge \mathbf{B})$

c) $\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (B \cdot \nabla)\mathbf{A} - (A \cdot \nabla)\mathbf{B}$

d) $(\mathbf{A} \wedge \nabla) \cdot \mathbf{B} = \mathbf{A} \cdot (\nabla \wedge \mathbf{B})$

Solution

Here we use that

$$(\mathbf{A} \wedge \mathbf{B})_i = \epsilon_{ijk}(\mathbf{A})_j(\mathbf{B})_k = \epsilon_{ijk}A_jB_k \qquad (6)$$

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{A})_i(\mathbf{B})_i = A_iB_i \qquad (7)$$

We also use the notation $\frac{\partial}{\partial x_i} = \partial_i$ and thus $(\nabla)_i = \partial_i$

a)

$$\begin{aligned}(\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}))_i &= \epsilon_{ijk}A_j(\mathbf{B} \wedge \mathbf{C})_k & (8)\\ &= \epsilon_{ijk}A_j(\epsilon_{kln}B_lC_n) = \epsilon_{ijk}\epsilon_{kln}A_jB_lC_n = \epsilon_{kij}\epsilon_{kln}A_jB_lC_n & (9)\\ &\overset{(4)}{=} (\delta_{il}\delta_{jn} - \delta_{in}\delta_{jl})A_jB_lC_n & (10)\\ &= A_jB_iC_j - A_jB_jC_i & (11)\\ &= (\mathbf{B})_i(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{C})_i(\mathbf{A} \cdot \mathbf{B}) & (12)\end{aligned}$$

b)

$$\begin{aligned}\nabla \cdot (\mathbf{A} \wedge \mathbf{B}) &= \partial_i(\epsilon_{ijk}A_jB_k) \overset{\text{Leibniz rule}}{=} (\epsilon_{ijk}\partial_i(A_j)B_k) + (\epsilon_{ijk}A_j\partial_i(B_k)) & (13)\\ &= \epsilon_{kij}B_k\partial_i(A_j) - \epsilon_{jik}A_j\partial_i(B_k) & (14)\\ &= B_k(\epsilon_{kij}\partial_iA_j) - A_j(\epsilon_{jik}\partial_iB_k) & (15)\\ &= (\nabla \wedge \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \wedge \mathbf{B}) & (16)\end{aligned}$$

c)

$$
\begin{aligned}
\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) &= \epsilon_{ijk} \nabla \wedge (A_i B_j \hat{e}_k) = && (17)\\
&= \epsilon_{ijk} \epsilon_{lkn} \partial_l (A_i B_j) \hat{e}_n = && (18)\\
&= \epsilon_{kij} \epsilon_{knl} \left[B_j \partial_l A_i + A_i \partial_l B_j \right] \hat{e}_n = && (19)\\
&\overset{(4)}{=} (\delta_{in} \delta_{jl} - \delta_{il} \delta_{jn}) \left[B_j \partial_l A_i + A_i \partial_l B_j \right] \hat{e}_n = && (20)\\
&= B_j \partial_j A_i \hat{e}_i + A_i \partial_j B_j \hat{e}_i - \partial_i A_i B_j \hat{e}_j - A_i \partial_i B_j \hat{e}_j = && (21)\\
&= (\mathbf{B} \cdot \nabla) \mathbf{A} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\nabla \cdot \mathbf{A}) \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B}. && (22)
\end{aligned}
$$

d)

$$
\begin{aligned}
(\mathbf{A} \wedge \nabla) \cdot \mathbf{B} &= (\epsilon_{ijk} A_j \partial_k) \hat{e}_i \cdot B_l \hat{e}_l = && (23)\\
&= \delta_{il} (\epsilon_{ijk} A_j \partial_k) B_l = && (24)\\
&= \epsilon_{ijk} A_j \partial_k B_i = \epsilon_{jki} A_j \partial_k B_i = && (25)\\
&= \mathbf{A} \cdot (\nabla \wedge \mathbf{B}) && (26)
\end{aligned}
$$

3. Gauge transformations

a) Show that the potentials

$$
\begin{aligned}
\phi &= 0 & \mathbf{A} &= \frac{B}{2}(-y, x, 0) && (27)\\
\phi' &= 0 & \mathbf{A}' &= B(0, x, 0) && (28)
\end{aligned}
$$

are equivalent and represent the same magnetic and electric fields. Find the gauge transformation that relates them. Here B is a constant and (x_1, x_2, x_3) represent the explicit components of a vector.

b) Show that the potentials

$$
\begin{aligned}
\phi &= -\mathbf{E}_0 \cdot \mathbf{r} \sin(\omega t) & \mathbf{A} &= \mathbf{0} && (29)\\
\phi' &= 0 & \mathbf{A}' &= \mathbf{E}_0 \frac{1}{\omega} \cos(\omega t) && (30)
\end{aligned}
$$

are equivalent and represent the same magnetic and electric fields. Find the gauge transformation that relates them.

Solution

a) We start with the magnetic field:

$$
\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} = (0, 0, B)\\
\mathbf{B}' &= \nabla \times \mathbf{A}' = (0, 0, B) \ . && (31)
\end{aligned}
$$

Now we compute the electric field:

$$
\begin{aligned}
\mathbf{E} &= -\nabla \phi - \frac{d\mathbf{A}}{dt} = 0\\
\mathbf{E}' &= -\nabla \phi' - \frac{d\mathbf{A}'}{dt} = 0 \ . && (32)
\end{aligned}
$$

A possible gauge transformation is $\chi(x,y,z) = \frac{B}{2}xy$. So we have

$$\nabla\chi = \frac{B}{2}(y,x,0)\ , \tag{33}$$

and

$$\mathbf{A} \to \mathbf{A} + \nabla\chi = \mathbf{A}'\ . \tag{34}$$

The electric potential does not change under this transformation:

$$\phi \to \phi - \frac{\partial\chi}{\partial t} = 0 = \phi'\ . \tag{35}$$

b) Again, we start by computing the electric and magnetic field to see if both potentials are equivalent.

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} = \mathbf{E_0}\sin(\omega t) \qquad \mathbf{B} = -\nabla\times\mathbf{A} = 0\ , \tag{36}$$

$$\mathbf{E}' = -\nabla\phi' - \frac{\partial\mathbf{A}'}{\partial t} = \mathbf{E_0}\sin(\omega t) \qquad \mathbf{B}' = -\nabla\times\mathbf{A}' = 0\ . \tag{37}$$

We see that both fields are equal, so there must be a gauge transformation relating the potentials. Such a gauge transformation χ would satisfy:

$$\phi' = \phi - \frac{\partial\chi}{\partial t}, \qquad \mathbf{A}' = \mathbf{A} + \nabla\chi\ . \tag{38}$$

The first equation implies

$$\chi = \mathbf{E}_0\cdot\mathbf{r}\frac{1}{\omega}\cos(\omega t) + f(\mathbf{r})\ , \tag{39}$$

and the second implies $\nabla f = 0$. Therefore, we can simply set $f(\mathbf{r}) = 0$.

4. Gradient in spherical coordinates Starting from the expression of the gradient in cartesian coordinates

$$(\nabla f)_{\text{cartesian}} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

derive the expression for the gradient in spherical coordinates

$$(\nabla f)_{\text{spherical}} = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right).$$

Solutions Let us use the coordinates x_i for the cartesian coordinate and y_i for the spherical. As usual

$$\begin{aligned} x_1 &= r\sin\theta\cos\phi = y_1\sin y_2\cos y_3 \\ x_2 &= r\sin\theta\sin\phi = y_1\sin y_2\sin y_3 \\ x_3 &= r\cos\theta = y_1\cos y_2 \end{aligned} \tag{40}$$

or the inverse relation

$$\begin{cases} r = y_1 = (x_1^2 + x_2^2 + x_3^2)^{1/2} = (x_i x_i)^{1/2} \\ \theta = y_2 = \arccos\left(\dfrac{x_3}{(x_i x_i)^{1/2}}\right) \\ \phi = y_3 = \arctan\left(\dfrac{x_2}{x_1}\right) \end{cases} \tag{41}$$

Now we know that $x_i(\mathbf{y})$, thus

$$\frac{\partial f}{\partial x_i} = \frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j} \tag{42}$$

Moreover, using the shorthand $\rho^2 = x_1^2 + x_2^2$, the versors transform according to the map

$$T : \{\hat{\mathbf{y}}_i\} \to \{\hat{\mathbf{x}}_i\} \quad \text{s.t.} \quad \begin{cases} \hat{\mathbf{x}}_1 = \frac{x_1}{r}\hat{\mathbf{y}}_1 + \frac{x_1}{\rho}\frac{x_3}{r}\hat{\mathbf{y}}_2 - \frac{x_2}{\rho}\hat{\mathbf{y}}_3; \\ \hat{\mathbf{x}}_2 = \frac{x_2}{r}\hat{\mathbf{y}}_1 + \frac{x_2}{\rho}\frac{x_3}{r}\hat{\mathbf{y}}_2 + \frac{x_1}{\rho}\hat{\mathbf{y}}_3; \\ \hat{\mathbf{x}}_3 = \frac{x_3}{r}\hat{\mathbf{y}}_1 - \frac{\rho}{r}\hat{\mathbf{y}}_2. \end{cases}$$

At this point we can write the gradient operator as

$$\begin{aligned} \nabla = \left(\nabla^{cart}\right)_i \hat{\mathbf{x}}_i &= \left(\frac{\partial}{\partial x_i}\right)\hat{\mathbf{x}}_i = \\ &= \left(\frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}\right)\hat{\mathbf{x}}_i = \\ &= \left(\frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial y_j}\right)T_{ik}\hat{\mathbf{y}}_k \end{aligned}$$

The $k-$th component of the gradient in spherical coordinates is then

$$T_{ik}\left(\frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}\right). \tag{43}$$

At this point we only have to substitute all the relevant pieces and perform the sums (over i and j, the repeated indices in the expression).

Let's derive the radial component for example: $k = 1$, and let's compute separately the part proportional to $\frac{\partial}{\partial y_1} = \frac{\partial}{\partial r}$, $\frac{\partial}{\partial y_2} = \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial y_3} = \frac{\partial}{\partial \phi}$.

- $\propto \frac{\partial}{\partial r}$: it's enough to set $k = 1$ and $j = 1$ in equation (43) (and sum over i).

$$\begin{aligned} &T_{11}\frac{\partial r}{\partial x_1} + T_{21}\frac{\partial r}{\partial x_2} + T_{31}\frac{\partial r}{\partial x_3} = \\ &= \frac{x_1}{r}\cdot\frac{x_1}{r} + \frac{x_2}{r}\cdot\frac{x_2}{r} + \frac{x_3}{r}\cdot\frac{x_3}{r} = 1 \end{aligned}$$

- $\propto \frac{\partial}{\partial \theta}$: corresponds to $k = 1$ and $j = 2$

$$\begin{aligned} &T_{11}\frac{\partial \theta}{\partial x_1} + T_{21}\frac{\partial \theta}{\partial x_2} + T_{31}\frac{\partial \theta}{\partial x_3} = \\ &\frac{x_1}{r}\cdot\frac{1}{\rho}\frac{x_1}{r} + \frac{x_2}{r}\cdot\frac{1}{\rho}\frac{x_2}{r} - \frac{x_3}{r}\cdot\frac{1}{x_3}\frac{\rho}{r} = \\ &= \frac{\rho}{r^2} - \frac{\rho}{r^2} = 0 \end{aligned}$$

- $\propto \frac{\partial}{\partial \phi}$: finally corresponds to $k = 1$ and $j = 3$

$$T_{11}\frac{\partial \phi}{\partial x_1} + T_{21}\frac{\partial \phi}{\partial x_2} + T_{31}\frac{\partial \phi}{\partial x_3} =$$
$$-\frac{x_1}{r} \cdot \frac{1}{\rho}\frac{x_2}{x_1} + \frac{x_2}{r} \cdot \frac{1}{\rho} + \frac{x_3}{r} \cdot 0 = 0.$$

So the radial component of the gradient in spherical coordinates is $\hat{\mathbf{r}} \cdot \nabla = \frac{\partial}{\partial r}$. All the other components can be computed analogously, by considering $k = 2, 3$.

Alternative solution

We first start from the expression of the gradient in cartesian basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and cartesian coordinates $(x, y, z, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ and will note $(x, y, z) = (x_1, x_2, x_3)$ too to make notation simpler:

$$\nabla f(x, y, z) = \frac{\partial f(x, y, z)}{\partial x}\mathbf{e}_x + \frac{\partial f(x, y, z)}{\partial y}\mathbf{e}_y + \frac{\partial f(x, y, z)}{\partial z}\mathbf{e}_z = \frac{\partial f}{\partial x_i}\mathbf{e}_i \qquad (44)$$

where the last equality uses Einstein's summation convention.

We want to both change basis to $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi) = (\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ and coordinates to $(r, \theta, \phi, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi})$ where will use $(r, \theta, \phi) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ again. The goal is now to express the same vector field in a different basis and in different coordinates:

$$\nabla f = D_1(f)\mathbf{e}_r + D_2(f)\mathbf{e}_\theta + D_3(f)\mathbf{e}_\phi = D_i(f)\tilde{\mathbf{e}}_i, \qquad (45)$$

where the D's are combinations of derivatives and coordinates $\tilde{x}_j$'s and $\frac{\partial}{\partial \tilde{x}_k}$'s. It should be reminded that, contrary to the cartesian basis, the spherical basis vectors depend on (r, θ, ϕ) and are not constant.

Let's start by a useful definition of the $\tilde{\mathbf{e}}_i$'s (no summation implied on the right-hand side):

$$\tilde{\mathbf{e}}_i(r, \theta, \phi) = \frac{\frac{\partial \mathbf{p}(r,\theta,\phi)}{\partial \tilde{x}_i}}{\|\frac{\partial \mathbf{p}(r,\theta,\phi)}{\partial \tilde{x}_i}\|}, \qquad (46)$$

where $\mathbf{p} = x_i\mathbf{e}_i$ is just the position vector. We could use this expression to compute explicitly each spherical basis vector in term of the cartesian basis vectors, but we';; take a different route here.

Since $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ form a orthonormal basis, we can find each components of any vector expressed in that basis by taking the dot product: $\mathbf{v} = \tilde{v}_i\tilde{\mathbf{e}}_i \Rightarrow \tilde{v}_i = \mathbf{v} \cdot \tilde{\mathbf{e}}_i$. Thus, we get

$$D_i(f) = (\nabla f) \cdot \tilde{\mathbf{e}}_i = \frac{\partial f}{\partial x_j}\mathbf{e}_j \cdot \tilde{\mathbf{e}}_i = \frac{\partial f}{\partial x_j}\mathbf{e}_j \cdot \left(\frac{\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}}{\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\|}\right) \qquad (47)$$

$$= \frac{1}{\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\|}\frac{\partial f}{\partial x_j}\mathbf{e}_j \cdot \frac{\partial (x_k\mathbf{e}_k)}{\partial \tilde{x}_i}. \qquad (48)$$

Now, because $\mathbf{e}_k$ are constant, we can pull them out of the derivative, and by the linearity of the dot product, we get

$$D_i(f) = \frac{1}{\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\|}\frac{\partial x_k}{\partial \tilde{x}_i}\frac{\partial f}{\partial x_j}(\mathbf{e}_j \cdot \mathbf{e}_k). \qquad (49)$$

We finally use $(\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk})$ to get

$$D_i(f) = \frac{1}{\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\|}\frac{\partial x_k}{\partial \tilde{x}_i}\frac{\partial f}{\partial x_j}\delta_{jk} = \frac{1}{\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\|}\frac{\partial x_j}{\partial \tilde{x}_i}\frac{\partial f}{\partial x_j} = \frac{1}{\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\|}\frac{\partial f}{\partial \tilde{x}_i}, \tag{50}$$

where in the last equality we use the multivariable chain rule. Finally we need to compute the normalization factors. We can do that in cartesian basis: $\mathbf{v} = v_i\mathbf{e}_i \Rightarrow \|\mathbf{v}\| = \sqrt{v_iv_i}$, and so, using $\mathbf{p} = x_j\mathbf{e}_j$, the normalization factors are given by

$$\left\|\frac{\partial \mathbf{p}}{\partial \tilde{x}_i}\right\| = \sqrt{\sum_j \left(\frac{\partial x_j}{\partial \tilde{x}_i}\right)^2}, \tag{51}$$

which leads to

$$\nabla f = \frac{1}{\sqrt{\sum_j \left(\frac{\partial x_j}{\partial \tilde{x}_i}\right)^2}}\frac{\partial f}{\partial \tilde{x}_i}\tilde{\mathbf{e}}_i. \tag{52}$$

5. Dirac δ-function

a) Show that

$$\lim_{\alpha\to 0^+}\frac{\alpha}{\pi(\alpha^2 + x^2)} \tag{53}$$

is a representation of the δ-function by verifying that $\int_{-\infty}^{\infty} dx\delta(x)f(x) = f(0)$, where f is a smooth test function that does not grow at infinity.

b) Prove the following identity

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)\ , \tag{54}$$

where the sum is over all the zeros $f(x_i) = 0$ and we assume that $f'(x_i) \neq 0$.

c) Prove the following identity in $\mathbb{R}^n$

$$\int d^n\mathbf{x} f(\mathbf{x})\nabla_{\mathbf{x}}\delta^n\left(\mathbf{x} - \mathbf{x}_0\right) = -\left.\nabla_{\mathbf{x}} f\right|_{\mathbf{x}=\mathbf{x}_0}\ . \tag{55}$$

d) Let $g(x)$ be a bounded smooth function. Compute $\int dxg(x)\theta'(x - x_0)$ and derive a relation between the Heaviside θ-function and the Dirac δ-function.

e) Evaluate the following integrals

$$\int dxg(x)\delta'(x - x_0)\ , \qquad \int dxg(x)\delta''(x - x_0)\ . \tag{56}$$

f) Calculate $\exp\left(x_0\frac{d}{dx}\right)\delta(x)$.

g) Show that $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}$.

Solution

a) In exercises of this kind it is very important to remember that the integral must be evaluated *before* the limit. We must solve the following integral:

$$I(\alpha) = \int_{-\infty}^{+\infty} dx \, \frac{\alpha}{\pi(\alpha^2 + x^2)} f(x) \; . \tag{57}$$

We can do this by extending the integral to the complex plane. Indeed, we have

$$I(\alpha) = \lim_{R \to \infty} \oint_{\gamma(R)} dz \, \frac{\alpha}{\pi(\alpha^2 + z^2)} f(z) \; , \tag{58}$$

where $\gamma(R)$ is a semi-circle in the complex plane with a diameter which goes from $-R$ to R. After the integral is calculated, R is sent to infinity. We choose to write the semi-circle in the positive semi-plane[1]. The reason why equation (58) is legitimate is the behavior of the integrand for $|z| \to \infty$: it goes to zero faster than $1/|z|$ in every direction. Inside the contour $\gamma(R)$ there is one pole at $z = i\alpha$ plus other possible poles of the function f at $z = z_i$. The integral can be evaluated with the method of residues:

$$I(\alpha) = 2\pi i \sum \mathrm{Res} \frac{\alpha}{\pi(\alpha^2 + z^2)} f(z) = f(i\alpha) + \sum_j \frac{2i\alpha}{\alpha^2 + z_j^2} \mathrm{Res} f(z_j) \; . \tag{59}$$

Therefore, we can see that

$$\int_{-\infty}^{+\infty} dx \, \lim_{\alpha \to 0} \frac{\alpha}{\pi(\alpha^2 + x^2)} f(x) = \lim_{\alpha \to 0} f(i\alpha) = f(0) = \int_{-\infty}^{\infty} dx \, \delta(x) \, f(x) \; . \tag{60}$$

b) <u>First method</u>: In order to prove the identity, we must understand what happens when we apply $\delta(f(x))$ to a test function $h(x)$. More precisely, we must find the result of

$$I = \int_{-\infty}^{+\infty} dx \, \delta(f(x)) \, h(x) \; . \tag{61}$$

We know that the delta function is different from zero only when its argument is zero, therefore we can write the integral as a sum of contributions in the proximity of the zeros of f:

$$I = \sum_i \int_{x_i - \epsilon}^{x_i + \epsilon} dx \, \delta(f(x)) \, h(x) \equiv \sum_i I_i \; , \tag{62}$$

where ϵ is a positive infinitesimal parameter. Consider a single term I_i. We do the change of variables

$$\begin{aligned} x &\to y \equiv f(x) \; , \\ x_i \pm \epsilon &\to f(x_i \pm \epsilon) \simeq \pm\epsilon f'(x_i) \; , \\ dx &= \frac{df^{-1}}{dy}(y) dy \; . \end{aligned} \tag{63}$$

[1] in this particular case, both the semi-planes were possible.

The integral thus becomes

$$
\begin{aligned}
I_i = & \int_{-\epsilon f'(x_i)}^{+\epsilon f'(x_i)} dy\, \delta(y)\, h(f^{-1}(y))\, \frac{df^{-1}}{dy}(y) = \\
= & \pm h(f^{-1}(0)) \frac{df^{-1}}{dy}(0)\ .
\end{aligned}
\tag{64}
$$

We have written " $\pm$ " because the sign is positive if $f'(x_i) > 0$ and negative otherwise[2]. We must now remember the theorem of the inverse function: if $y = g(x)$ is a monotonic differentiable function, in a point x_1 so that $g'(x_1) \neq 0$, the following equality holds:

$$
\left[\frac{dg^{-1}}{dy}\right]_{y=g(x_1)} = \frac{1}{g'(x_1)}\ .
\tag{65}
$$

Using this theorem, and the fact that $f^{-1}(0) = x_i$, we can write

$$
I_i = \pm h(x_i) \frac{1}{f'(x_i)} = h(x_i) \frac{1}{|f'(x_i)|}\ .
\tag{66}
$$

Collecting all the results, we have:

$$
\begin{aligned}
I = & \int_{-\infty}^{+\infty} dx\, \delta(f(x))\, h(x) = \\
= & \sum_i h(x_i) \frac{1}{|f'(x_i)|} = \\
= & \int_{-\infty}^{+\infty} dx \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}\, h(x)\ .
\end{aligned}
\tag{67}
$$

Comparing the first and the third lines of (67) we can deduce that

$$
\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}\ .
\tag{68}
$$

<u>Second method</u>: Even in this case, we must treat separately the neighborhood of each zero x_i. We use the definition of the Dirac δ of the previous exercise and Taylor expand the function f around each zero x_i:

$$
\begin{aligned}
\lim_{\alpha \to 0} \frac{\alpha}{\pi(\alpha^2 + (f(x))^2)} \simeq & \lim_{\alpha \to 0} \frac{\alpha}{|f'(x_i)|} \frac{1}{|f'(x_i)| \pi \left(\left(\frac{\alpha}{f'(x_i)} \right)^2 + (x - x_i)^2 \right)} \\
= & \lim_{\beta \to 0} \frac{\beta}{\pi(\beta^2 + (x - x_i)^2)} \frac{1}{|f'(x_i)|} \\
= & \delta(x - x_i) \frac{1}{|f'(x_i)|}\ .
\end{aligned}
\tag{69}
$$

When we sum over i we obtain

$$
\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}\ .
\tag{70}
$$

[2]Indeed, from the definition of the Dirac δ, we can deduce that $\int_{+\infty}^{-\infty} dx \delta(x) g(x) = -g(0)$.

c) We treat each component i of the gradient vector separately. Suppose that the interval of the integral for the coordinate x_i is $\zeta = (a_i, b_i)$, where a_i and b_i can be either finite or infinite. We integrate by parts:

$$\begin{aligned}
&\int d^n\mathbf{x} f(\mathbf{x})\partial_i\delta^n\left(\mathbf{x}-\mathbf{x}_0\right) = \\
=&\int d^{n-1}\mathbf{x}\left[f(\mathbf{x})\delta^n\left(\mathbf{x}-\mathbf{x}_0\right)\right]_{b_i}^{a_i} - \int d^n\mathbf{x}\partial_i f(\mathbf{x})\delta^n\left(\mathbf{x}-\mathbf{x}_0\right) = \\
=&-\left.\partial_i f\right|_{\mathbf{x}=\mathbf{x}_0} .
\end{aligned} \tag{71}$$

We have put equal to zero the first term in the second line of (71) because the δ function is zero for $x_i \neq x_{i0}$, and this is the case when $x_i = a_i$ and $x_i = b_i$. This is valid for all i so in vectorial form:

$$\int d^n\mathbf{x} f(\mathbf{x})\nabla_{\mathbf{x}}\delta^n\left(\mathbf{x}-\mathbf{x}_0\right) = -\left.\nabla_{\mathbf{x}} f\right|_{\mathbf{x}=\mathbf{x}_0} . \tag{72}$$

d) As in the previous case, we need an integration by parts:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dx\, f(x)\partial_x\theta(x-x_0) = \\
=&\lim_{R\to\infty}\left[f(x)\theta(x-x_0)\right]_{-R}^{+R} - \lim_{R\to\infty}\int_{x_0}^{+R} dx\,\partial_x f(x) = \\
=&\lim_{R\to\infty}\left[f(+R) - f(+R) + f(x_0)\right] = \\
=&f(x_0) .
\end{aligned} \tag{73}$$

Therefore, the derivative of the Heaviside function is the Dirac δ.

e) By integration by parts, we have:

$$\int dx g(x)\delta'(x-x_0) = -\int dx g'(x)\delta(x-x_0) = -g'(x_0) . \tag{74}$$

In the first equality the boundary term vanishes because $g(x)\delta(x-x_0)$ goes to zero at infinity, the second equality comes from the definition of the Dirac δ function.

For the second integral, two successive integrations by parts gives us:

$$\int dx g(x)\delta''(x-x_0) = -\int dx g'(x)\delta'(x-x_0) = \int dx g''(x)\delta(x-x_0) = g''(x_0) . \tag{75}$$

f) Let's take a smooth test function f:

$$\begin{aligned}
\int dx f(x)\exp\left(x_0\frac{d}{dx}\right)\delta(x) &= \int dx f(x)\sum_{n=0}^{\infty}\frac{x_0^n}{n!}\frac{d^n\delta(x)}{dx^n} &(76)\\
&= \sum_{n=0}^{\infty}\frac{x_0^n}{n!}\int dx f(x)\frac{d^n\delta(x)}{dx^n} \\
&= \sum_{n=0}^{\infty}\frac{x_0^n}{n!}(-1)^n f^{(n)}(0) = \sum_{n=0}^{\infty}\frac{(-x_0)^n}{n!}f^{(n)}(0) \\
&= f(-x_0) . &(77)
\end{aligned}$$

The first equality comes from the definition of the exponential and the third from an immediate generalisation of the results obtained in question e). The last line comes from the Taylor expansion formula. This is valid for all smooth test functions so we can conclude:

$$\exp\left(x_0\frac{d}{dx}\right)\delta(x) = \delta(x+x_0)\ . \tag{78}$$

g) First method: by Fourier transform. We write f as the anti-transform of its Fourier transform:

$$\begin{aligned} f(t_2) &= \int_{-\infty}^{+\infty} d\omega e^{i\omega t_2}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dt_1 e^{-i\omega t_1} f(t_1) = \\ &= \int_{-\infty}^{+\infty} dt_1 \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega(t_2-t_1)}\right] f(t_1)\ . \end{aligned} \tag{79}$$

Now, compare the term in square brackets in (79) with a delta function:

$$f(t_2) = \int_{-\infty}^{+\infty} dt_1 f(t_1)\delta(t_2-t_1)\ . \tag{80}$$

Therefore, we can write

$$\delta(t_2-t_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega(t_2-t_1)}\ , \tag{81}$$

and, in particular, when $t_1 = 0$,

$$\delta(t_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega e^{i\omega t_2}\ . \tag{82}$$

Second method: we use a regulator to compute the integral explicitly.

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} e^{-\epsilon^2\omega^2} \qquad (83) \\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\frac{\sqrt{\pi}}{\epsilon}\exp\left(-\frac{t^2}{4\epsilon^2}\right) = \lim_{\epsilon\to 0}\frac{1}{\sqrt{4\pi\epsilon^2}}\exp\left(-\frac{t^2}{4\epsilon^2}\right)\ . \end{aligned}$$

which is a representation of the Dirac delta. This can be checked as follows: let $f(x)$ be a smooth test function admitting a Taylor expansion in $x = 0$. The statement we need to show is

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dt\,\frac{1}{\sqrt{4\pi\epsilon^2}} f(t)\exp\left(-\frac{t^2}{4\epsilon^2}\right) = f(0)\ . \tag{84}$$

We proceed by splitting the integral into two regions: one close to $t = 0$ (say at distance δ) and its complement

$$\lim_{\epsilon\to 0}\left(\int_{|t|\leq\delta} + \int_{|t|>\delta}\right) dt\,\frac{1}{\sqrt{4\pi\epsilon^2}} f(t)\exp\left(-\frac{t^2}{4\epsilon^2}\right)\ . \tag{85}$$

The second integral has limit 0 for any fixed δ when $\epsilon \to 0$. While in the first one, we can choose δ so that $f(t)$ can be Taylor expanded.

$$\lim_{\epsilon\to 0} \frac{1}{\sqrt{4\pi}} \int_{|t|\leq\delta} dt\, \frac{1}{\epsilon} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \exp\left(-\frac{t^2}{4\epsilon^2}\right) \,. \tag{86}$$

Here actually the terms for odd n are zero. After a simple change of variables we can recast this expression in a way that the limit is easy to take

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{1}{\sqrt{4\pi}} \int_{|t|\leq\delta/\epsilon} dt \sum_{n=0}^{\infty} \epsilon^{2n} \frac{f^{(2n)}(0)}{(2n)!} t^{2n} e^{-t^2/4} = \\ = f(0)\, \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} dt\, e^{-t^2/4} \,. \end{aligned} \tag{87}$$

A simple computation shows that the result is indeed $f(0)$.

Third method: by using another type of regulator. First, we separate the integral into two parts

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} = \frac{1}{2\pi} \int_{-\infty}^{0} d\omega\, e^{i\omega t} + \frac{1}{2\pi} \int_{0}^{\infty} d\omega\, e^{i\omega t} \tag{88}$$

Next, with the intention of suppressing the integrands at $\omega \to \pm\infty$, we shift t to $t - i\epsilon$ in the first integral and to $t + i\epsilon$ in the second integral. Doing so allows us to compute the integrals explicitly

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} &= \frac{1}{2\pi} \lim_{\epsilon\to 0} \left[\int_{-\infty}^{0} d\omega\, e^{i\omega t + \epsilon\omega} + \int_{0}^{\infty} d\omega\, e^{i\omega t - \epsilon\omega} \right] \\ &= \frac{1}{2\pi} \lim_{\epsilon\to 0} \left[\frac{1}{\epsilon + it} + \frac{1}{\epsilon - it} \right] \\ &= \lim_{\epsilon\to 0} \frac{\epsilon}{\pi(\epsilon^2 + t^2)} \end{aligned} \tag{89}$$

which is exactly the representation of the δ-function we introduced in problem **2.a)**, and thus it follows that $\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} = \delta(t)$.