Classical Electrodynamics

Solutions week 1

1. Consider an infinite cylinder of radius R with a uniform charge density ρ within its volume. Denote its linear charge density by κ.

a) Express κ in terms of ρ and give the units of these two quantites.

b) Using the symmetry of the problem, compute the electric field $\mathbf{E}$ using Gauss's law and deduce the scalar potential ϕ.

c) Find the scalar potential ϕ by solving Poisson's equation.

Solution

a) ρ is the quantity of charge per unit volume given in $\frac{C}{m^3}$. κ is a charge per unit of lenght $\frac{C}{m}$. Now, if we cut a piece of the cylinder of height h it has charge

$$Q = \rho\pi R^2 h = \kappa h, \tag{1}$$

from which we deduce

$$\rho\pi R^2 = \kappa. \tag{2}$$

b) We want to find the electric field at a distance r from the axis of the cylinder. Consider an imaginary cylinder C of radius r and height h with the same axis as the cylinder of radius R. Let first assume $r > R$. Using Gauss's law, we have that the flux of the electric field is

$$\Phi(E) = \frac{Q}{\epsilon_0} = \frac{\kappa h}{\epsilon_0} \ . \tag{3}$$

By looking at the symmetry of the problem, it is clear that the electric field is orthogonal to the lateral surface of the of the imaginary cylinder C : $\mathbf{E} = E\hat{e}_r$. On the other hand it is parallel to the top and bottom faces of C. Therefore, we have

$$\Phi(E) = \int dS \ \mathbf{E}\cdot\mathbf{n} = 2\pi r h E \ . \tag{4}$$

If we now plug (4) into (3), we obtain

$$\mathbf{E} = \frac{\kappa}{2\pi r\epsilon_0}\hat{e}_r \ . \tag{5}$$

In cylindrical coordinate, the gradient of a function f is given by

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\varphi}\hat{e}_\varphi + \frac{\partial f}{\partial z}\hat{e}_z \ . \tag{6}$$

We know that in electrostatics the relation between the potential and the electric field is

$$\nabla\phi = -\mathbf{E} \ , \tag{7}$$

therefore the potential is the solution of the equation

$$\frac{\partial \phi}{\partial r} = -\frac{\kappa}{2\pi r \epsilon_0} \ , \tag{8}$$

which is

$$\phi(r) = -\frac{\kappa}{2\pi\epsilon_0} \ln\left[\frac{r}{R}\right] + C_1 \ , \tag{9}$$

where C_1 is a constant that still has to be determined. We must now solve the problem in the case $r < R$. Using again Gauss's law, it is possible to show that only the portion of the charge that is inside the cylinder of radius r contributes to the electric field in a point at distance r from the axis. The flux is

$$\Phi(\mathbf{E}) = E2\pi r h = \frac{\kappa h \pi r^2}{\pi \epsilon_0 R^2} = \frac{\kappa h r^2}{\epsilon_0 R^2} \ , \tag{10}$$

and the electric field is

$$\mathbf{E} = \frac{\kappa r}{2\pi\epsilon_0 R^2}\hat{e}_r \ . \tag{11}$$

The potential is obtained in the same way as above and is given by the following expression:

$$\phi = -\frac{\kappa r^2}{4\pi\epsilon_0 R^2} + C_2 \ . \tag{12}$$

Now we need to properly choose the two constants in order to have the continuity between the two regimes. A possible choice is $C_2 = 0$ and $C_1 = -\frac{\kappa}{4\pi\epsilon_0}$.

c) The Poisson equation for $r < R$ is

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} = -\frac{\kappa}{\pi R^2 \epsilon_0} \ . \tag{13}$$

If we write this equation in cylindrical coordinates, we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \phi}{\partial r}\right] = -\frac{\kappa}{\pi R^2 \epsilon_0} \ . \tag{14}$$

In equation (14) we have used the fact that ϕ depends only on r, by symmetry. After the first integration we obtain

$$r\frac{\partial \phi}{\partial r} = -\frac{\kappa r^2}{2\pi R^2 \epsilon_0} + C_1 \ . \tag{15}$$

With a second integration we get

$$\phi = -\frac{\kappa r^2}{4\pi R^2 \epsilon_0} + C_1 \ln\left[\frac{r}{R}\right] + C_2 \ . \tag{16}$$

We must choose $C_1 = 0$ in order to avoid non physical divergences for $r \to 0$. If instead $r > R$, the Poisson equation becomes

$$\nabla^2\phi = 0 \ , \tag{17}$$

which is, in cylindrical coordinates

$$\frac{\partial \phi}{\partial r} = \frac{C_3}{r} \ , \tag{18}$$

and therefore

$$\phi = C_3 \ln\left[\frac{r}{R}\right] + C_4 \tag{19}$$

Again, we must choose the proper values of the constants. A possible choice is $C_2 = 0$ and $C_4 = -\frac{\kappa}{4\pi\epsilon_0}$. In order to find C_3, we must impose the continuity of the electric field, that is satisfied when

$$\frac{\partial \phi_i}{\partial r}(R) = \frac{\partial \phi_e}{\partial r}(R) \tag{20}$$

where ϕ_i indicates the potential when $r \leq R$ and ϕ_e indicates the potential when $r \geq R$. In this way, we find $C_3 = -\frac{\kappa}{2\pi\epsilon_0}$

2. Consider a sphere of radius R with constant electric charge density $\rho > 0$.

a) Show that, both inside and outside of the sphere, the electric field is a power law function of the distance from the origin, i.e. it is proportional to r^n for some n. Find n for each region. Compute the electric potential, and plot the radial dependence of the electric field and potential.

b) Now assume there is a very narrow tunnel inside the sphere passing through the centre of the sphere. At time $t = 0$, a single point-like charge with electric charge $-q < 0$ and mass m is placed at rest in the tunnel at $r = a < R$. Neglecting the effect of radiation ($v \ll c$), find the equation of motion of this particle and solve it.

c) Two spheres, each of radius R and carrying uniform charge densities ρ and $-\rho$, respectively, are placed so that they partially overlap. Call $\mathbf{d}$ the vector from the positive centre to the negative centre, with $|\mathbf{d}| < 2R$. Show that the field in the region of overlap is constant and find its value.

Solution

a) Since the set up is spherically symmetric, the electric field is in $\hat{e}_r$ direction. Using the Gauss law for the outside of the sphere:

$$\int dS\ \vec{E} \cdot \vec{n} = E(4\pi r^2) = \frac{Q_{total}}{\epsilon_0} = \frac{4\pi R^3 \rho}{3\epsilon_0} \tag{21}$$

$$\vec{E} = \frac{\rho}{3\epsilon_0}\frac{R^3}{r^2}\,\hat{e}_r \qquad\qquad r \geq R \tag{22}$$

Similarly for the inside we have:

$$\int dS\ \vec{E} \cdot \vec{n} = E(4\pi r^2) = \frac{Q_r}{\epsilon_0} = \frac{4\pi r^3 \rho}{3\epsilon_0} \tag{23}$$

$$\vec{E} = \frac{\rho}{3\epsilon_0} r\,\hat{e}_r \qquad\qquad r \leq R \tag{24}$$

Therefore for the outside, $n = -2$ which means you can subsitute the sphere with a point-like charge. Inside the electric field grows linearly $n = 1$ which for a test charge would be like a spring as you will see in part b.

The potential is defined as

$$\vec{E} = -\nabla\phi \quad \Rightarrow \quad E_r = -\frac{\mathrm{d}\phi}{\mathrm{d}r}\,. \tag{25}$$

Integrating this equation in the two regions yields

$$\phi(r) = \begin{cases} -\frac{\rho r^2}{6\epsilon_0} + C_1 & r \le R\,, \\ \frac{\rho R^3}{3\epsilon_0 r} + C_2 & r \ge R\,. \end{cases} \tag{26}$$

One of the integration constants can be fixed by requiring the continuity of ϕ at $r = R$. This yields

$$C_1 = C_2 + \frac{\rho R^2}{2\epsilon_0}\,, \tag{27}$$

so finally

$$\phi(r) = \begin{cases} \frac{\rho(3R^2 - r^2)}{6\epsilon_0} + C_2 & r \le R\,, \\ \frac{\rho R^3}{3\epsilon_0 r} + C_2 & r \ge R\,. \end{cases} \tag{28}$$

b) As shown above, the electric field is linear in r and so is the force on the point-like charge:

$$\vec{F} = m\vec{a} = -q\vec{E} \tag{29}$$

$$m\ddot{r} + q\frac{\rho}{3\epsilon_0}r = 0 \tag{30}$$

which with the mentioned initial conditions has the solution:

$$r = a\,\cos(\omega t) \tag{31}$$

in which $\omega = \sqrt{q\frac{\rho}{3\epsilon_0 m}}$.

c) As solved in the previous question, inside a sphere with uniform charge densities, the electric field is proportional to the distance from the center. Let's note $\vec{x_1}$ and $\vec{x_2}$ the position of the center of the spheres and $\vec{r_1} = \vec{r} - \vec{x_1}$ and $\vec{r_2} = \vec{r} - \vec{x_2}$ the position relative to each center. In the intersection the total electric field will be:

$$\vec{E} = \frac{\rho}{3\epsilon_0}\vec{r_1} - \frac{\rho}{3\epsilon_0}\vec{r_2} = \frac{\rho}{3\epsilon_0}\vec{d} \tag{32}$$

where $\vec{d} = \vec{x_1} - \vec{x_2}$ is the distance between the spheres. Notice the result is constant in the intersection region.

3. A mass spectrometer is an instrument used to analyse the chemical composition of a material. The essential parts of a mass spectrometer are depicted in figure 1. At one end of the spectrometer, the material is heated so that some atoms are ionized, then these ions are accelerated by an electrostatic potential V, deflected by a magnetic field B and detected at the other end of the spectrometer. In practice, one varies the magnetic field B and measures the electric current carried by the ion beam that hits the electrode X.

a) For an ion of charge q and mass m, compute the magnetic field B necessary for the ion to hit the electrode X. Neglect the initial thermal velocity of the ion.

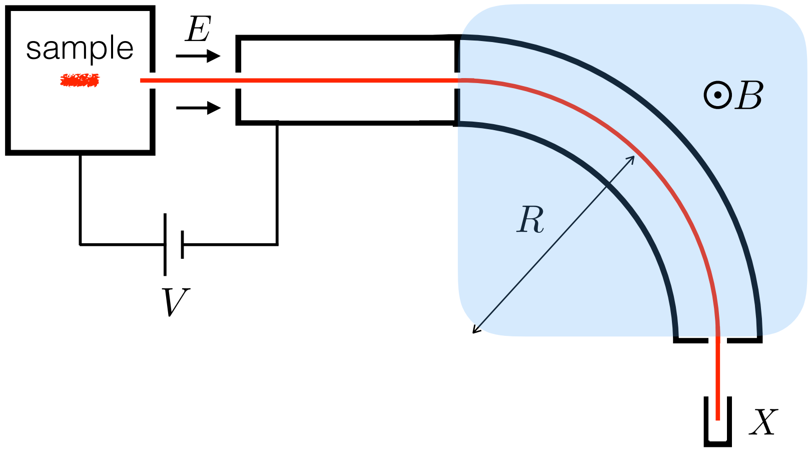

Figure 1: Schematic representation of a mass spectrometer. The ions are deflected by the magnetic field B perpendicular to the plane of the image. In order to hit the electrode X the ions have to follow a circular trajectory of radius R in the region with magnetic field (light blue region).

b) In the analysis of a pure substance in a spectrometer with potential $V = 1\,kV$ and $R = 35\,cm$, the magnetic field required to observe an electric current at X was $B = 98\,mT$. What was the substance?

c) **Challenge**: The ions created by thermal heating have random initial velocities. This will introduce some uncertainty in the measurement of the ratio q/m using the mass spectrometer. Can you estimate this uncertainty? Can you think of a strategy to select the initial velocity of the ions and reduce this uncertainty?

Solution

a) First, one has to find the velocity of the ion after the electric field region. Let's call this velocity v_e. We use energy conservation: at the beginning, the ion has zero velocity and thus has only potential energy qV. At the end $V = 0$ but the kinetic energy is $E_c = \frac{1}{2}mv_e^2$. Equating the two expressions, we get:

$$v_e = \sqrt{\frac{2qV}{m}}\ . \tag{33}$$

Then the ion enters the magnetic field region with a velocity orthogonal to the magnetic field. So it has a circular trajectory, of radius

$$r = \frac{v_e m}{qB}\ . \tag{34}$$

(To get this, you can write that the radial magnetic force qv_eB is equal to the mass times the acceleration in a circular moving frame $m\frac{v_e^2}{R}$ by Newton's law). In order for the ion to hit X, one must have $r = R$. Solving for B, one gets:

$$B = \frac{1}{R}\sqrt{\frac{2mV}{q}}\ . \tag{35}$$

b) What we actually measure is the ratio $\frac{m}{q}$. Inverting the previous equation:

$$\frac{m}{q} = \frac{R^2B^2}{2V} \tag{36}$$

and putting in the numbers, *in S.I. units*:

$$\frac{m}{q} = \frac{R^2B^2}{2V} = 5.88 \times 10^{-7}\,\mathrm{kg} \cdot \mathrm{C}^{-1} \ . \tag{37}$$

But the moving charge is an ion, so atomic units are more appropriate. In terms of the electron charge $e = 1.602 \times 10^{-19}$ C and the atomic mass unit $u = 1.661 \times 10^{-27}$ kg :

$$\frac{m}{q} = \frac{R^2B^2}{2V} = 56.7\,\mathrm{u} \cdot e^{-1} \ . \tag{38}$$

An ion has a charge of an integer times e. From the direction of the magnetic field, we can see that the ion is positive. If it has a charge +1, it has a mass of 56.7 u, corresponding to ^{56}Fe (or ^{57}Fe, but this is more rare), and iron has a +1 ion. If it has a charge +2, it has a mass of 113.5 u, in that case it would correspond to ^{114}Cd or ^{114}Sn, which both exists in a +2 state. In this case, we should also be able to observe the +1 ions by increasing the magnetic field by a factor of $\sqrt{2}$.

Most likely the ion is a $^{56}Fe^{+}$.

c) The particles of a substance at temperature T, assuming they are free, will follow the Maxwell-Boltzmann distribution:

$$p(v)dv \propto v^2 e^{-\frac{mv^2}{2k_bT}} dv \ . \tag{39}$$

where $p(v)dv$ is the probability of finding a particle with speed between v and $v + dv$, k_b is the Boltzmann constant and m is the mass of a particle. The exponential factor in equation (39) ensures that the uncertainty in v will be of order σ with

$$\sigma^2 = \frac{k_bT}{m} \ . \tag{40}$$

Here we are considering a one dimensional problem because the ions with non-zero transverse velocities due to thermal fluctuations are filtered by the aperture to the region with magnetic field. Since faster particles escape the hole more rapidly, the distribution of the escaped particles will favour higher speeds (with $p(v_x) \propto v_x e^{-\frac{mv_x^2}{2k_bT}}$). Nevertheless, as long as the exponential factor is present, the uncertainty in the longitudinal velocity v_x will be of the order of σ given in equation (40). From the previous results, we can propagate the uncertainty over v to the ratio m/q.

Equation (33) is modified into

$$v_e = \sqrt{\frac{2qV}{m} + v^2} \, . \tag{41}$$

Thus we can again derive from (34)

$$R^2B^2 = \frac{m^2v_e^2}{q^2} = \frac{m}{q}\left(2V + \frac{mv^2}{q}\right) \tag{42}$$

$$\Rightarrow \frac{m}{q} = \frac{R^2B^2}{2\left(V + \frac{\frac{1}{2}mv^2}{q}\right)} \simeq \frac{R^2B^2}{2V}\left(1 - \frac{mv^2}{2qV}\right) \ . \tag{43}$$

From which it follows

$$\eta \equiv \frac{\Delta\left(m/q\right)}{m/q} = \frac{m}{2qV}\sigma^2 = \frac{k_bT}{2qV} \ . \tag{44}$$

If we assume that, after ionization, the ion has an energy of the order of a few eV , and that the potential energy is $qV = 10^3 eV$, we have $\eta \sim$ few %.

In order to select the velocity, ions are submitted to a magnetic field $\mathbf{B}_1$ and an electric field $\mathbf{E}_1$, perpendicular to each other and to the direction of the ion, after being accelerated by the potential V. In this way, we have

$$q\mathbf{E}_1 + q\mathbf{v} \times \mathbf{B}_1 = m\mathbf{a} \ . \tag{45}$$

By properly choosing $\mathbf{E}_1$ and $\mathbf{B}_1$ it is possible to select a particular velocity $v_1 = \frac{E_1}{B_1}$, the only velocity at which the particles are not deflected. For instance, if $\mathbf{v} = v(0,0,1)$, we can set $\mathbf{E}_1 = E_1(1,0,0)$ and $\mathbf{B}_1 = B_1(0,1,0)$, so that

$$m\mathbf{a} = qE_1(1,0,0) - qvB_1(1,0,0) \ , \tag{46}$$

and when $v = \frac{E_1}{B_1}$ we have $\mathbf{a} = 0$.

4. Scalar and vector potentials.

a) Show that a vector field $\mathbf{V}$ obeying $\nabla \times \mathbf{V} = 0$ can be written as $\mathbf{V} = -\nabla\phi$, for some scalar potential ϕ. Check that this applies to the electric field $\mathbf{E}$ in electrostatics.

b) Similarly, show that a vector field $\mathbf{V}$ obeying $\nabla \cdot \mathbf{V} = 0$ can be written as $\mathbf{V} = \nabla \times \mathbf{A}$, for some vector potential $\mathbf{A}$. Check that this applies to the magnetic field $\mathbf{B}$ in magnetostatics.

c) Show that a general vector field $\mathbf{V}$ can be written as a sum $\mathbf{V} = \mathbf{V}_\parallel + \mathbf{V}_\perp$, with $\nabla \times \mathbf{V}_\parallel = 0$ and $\nabla \cdot \mathbf{V}_\perp = 0$.

Solution

a) Consider the relation[1] $(\nabla \times \mathbf{V}(\mathbf{x}))_k \equiv \partial_i V_j\, \epsilon_{ijk} = 0$. If we evaluate the Fourier transform, we obtain[2]

$$(\mathbf{k} \times \tilde{\mathbf{V}}(\mathbf{k}))_k \equiv k_i \tilde{V}_j\, \epsilon_{ijk} = 0 \ , \tag{47}$$

where $\mathbf{k}$ is the variable in Fourier space and $\tilde{\mathbf{V}}$ is the Fourier transform of $\mathbf{V}$. Equation (47) can be demonstrated with an integration by parts, using the hypothesis that V at infinity converges to a constant that we can set to 0:

$$\begin{aligned}\mathcal{F}\left[\partial_i V_j\, \epsilon_{ijk}\right] &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3x\, e^{ik_i x_i}\, \partial_i V_j\, \epsilon_{ijk} = \\ &= -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3x\, \partial_i e^{ik_i x_i}\, V_j\, \epsilon_{ijk} = \ = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3x\, e^{ik_i x_i}\, ik_i V_j\, \epsilon_{ijk} = \\ &= -ik_i \tilde{V}_j\, \epsilon_{ijk} \ ,\end{aligned} \tag{48}$$

The implication of (47) is that $\mathbf{k}$ and $\tilde{\mathbf{V}}$ are parallel. As a consequence, we can write $\tilde{\mathbf{V}}$ in the following way:

$$\tilde{\mathbf{V}}(\mathbf{k}) = i\mathbf{k}\tilde{\alpha}(\mathbf{k}) \qquad \text{for every } \mathbf{k} \tag{49}$$

[1]From now on, we will often use the Einstein notation, where $a_i b_i$ is a short notation for $\sum_i a_i b_i$.

[2]Here the letter i and k appear with two different meanings. Do not confuse k as an index referring to a spatial coordinate with the three dimensional variable k_i of the Fourier space. Also i as an index refers to spatial coordinates whereas i is the imaginary unit.

where $\tilde{\alpha}$ is a scalar. If we now anti-transform both sides of equation (49) we obtain:

$$V_i(\mathbf{x}) = \mathcal{F}^{-1}\left[ik_i\tilde{\alpha}(\mathbf{k})\right] = \int_{-\infty}^{\infty} d^3k\,(e^{-ik_jx_j}ik_i)\tilde{\alpha}(\mathbf{k})$$

$$= \int_{-\infty}^{\infty} d^3k\left(-\frac{\partial e^{-ik_jx_j}}{\partial x_i}\right)\tilde{\alpha}(\mathbf{k}) = -\frac{\partial}{\partial x_i}\mathcal{F}^{-1}\left[\tilde{\alpha}(\mathbf{k})\right] \tag{50}$$

$$= -\frac{\partial \alpha}{\partial x_i}(\mathbf{x}) \;, \tag{51}$$

where $\alpha(\mathbf{x})$ is the anti-transform of $\tilde{\alpha}(\mathbf{k})$. If we define $\phi = \alpha$ we have demonstrated that we can write the potential as $\mathbf{V} = -\nabla\phi$. If we identify $\mathbf{V}$ with the electric field, we can see that $\nabla\phi = -\mathbf{E}$ naturally follows from the Maxwell's equation $\nabla \times \mathbf{E} = 0$.

b) The relation $\nabla \cdot \mathbf{V} \equiv \partial_i V_i = 0$ becomes, under Fourier transform,

$$-i\mathbf{k}\cdot\tilde{\mathbf{V}} \equiv -ik_i\tilde{V}_i = 0 \;. \tag{52}$$

The derivation is similar to the one of the previous exercise:

$$\mathcal{F}\left[\partial_i V_i\right] = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3x\, e^{ik_ix_i}\,\partial_i V_i = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3x\,\partial_i e^{ik_ix_i}\,V_i = \tag{53}$$

$$= -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3x\, e^{ik_ix_i}\,ik_iV_i = -ik_i\tilde{V}_i \;.$$

The implication of (52) is that $-i\mathbf{k}$ and $\tilde{\mathbf{V}}$ are perpendicular. If two vectors are perpendicular, it is possible to write one as a cross product between the other and a certain vector $\tilde{\mathbf{A}}(\mathbf{k})$:

$$\tilde{\mathbf{V}} = -i\mathbf{k}\times\tilde{\mathbf{A}}(\mathbf{k}) \;. \tag{54}$$

We can now anti-transform on both sides equation (54) and obtain

$$V_l(\mathbf{x}) = \mathcal{F}^{-1}\left[-ik_i\tilde{A}_j(\mathbf{k})\epsilon_{ijl}\right] = -\int_{-\infty}^{\infty} d^3k\,(e^{-ik_jx_j}ik_i)\tilde{A}_j(\mathbf{k})\epsilon_{ijl} =$$

$$= -\int_{-\infty}^{\infty} d^3k\left(-\frac{\partial e^{-ik_jx_j}}{\partial x_i}\right)\tilde{A}_j(\mathbf{k})\epsilon_{ijl} = \frac{\partial}{\partial x_i}\mathcal{F}^{-1}\left[\tilde{A}_j(\mathbf{k})\right]\epsilon_{ijl} =$$

$$= \frac{\partial A_j}{\partial x_i}(\mathbf{x})\epsilon_{ijl} \;,$$

where $\mathbf{A}(\mathbf{x})$ is the anti-transform of $\tilde{\mathbf{A}}(\mathbf{k})$. For the case of magnetostatics, we must identify $\mathbf{V}$ with $\mathbf{B}$ and interpret $\mathbf{A}$ as the vector potential. Indeed, we can see that $\nabla\cdot\mathbf{B} = 0$ is one of the Maxwell's equation and we can write the magnetic field as the curl of the vector potential: $\mathbf{B} = \nabla\times\mathbf{A}$.

c) We consider, again, the Fourier transform $\tilde{\mathbf{V}}(\mathbf{k})$. The vector $\tilde{\mathbf{V}}$ can be decomposed in two components, one perpendicular to $\mathbf{k}$, that we call $\tilde{\mathbf{V}}_\perp$, and one parallel to $\mathbf{k}$, that we call $\tilde{\mathbf{V}}_\parallel$[3]. Let us call $\mathbf{V}_\perp$ and $\mathbf{V}_\parallel$ the anti-transforms of $\tilde{\mathbf{V}}_\perp$ and $\tilde{\mathbf{V}}_\parallel$ respectively. Using the previous results, we see that

$$\nabla\times\mathbf{V}_\parallel = 0 \;, \tag{56}$$

[3]Explicitly:

$$\tilde{\mathbf{V}}_\parallel = \frac{\tilde{\mathbf{V}}\cdot\mathbf{k}}{\mathbf{k}\cdot\mathbf{k}}\,\mathbf{k}, \qquad \tilde{\mathbf{V}}_\perp = \tilde{\mathbf{V}} - \tilde{\mathbf{V}}_\parallel. \tag{55}$$

and

$$\nabla \cdot \mathbf{V}_\perp = 0 \ . \tag{57}$$

We still have to prove that $\mathbf{V} = \mathbf{V}_\parallel + \mathbf{V}_\perp$. This can be done using the linearity of the inverse Fourier transform:

$$\begin{aligned} \mathbf{V} &= \mathcal{F}^{-1}\left[\tilde{\mathbf{V}}\right] = \mathcal{F}^{-1}\left[\tilde{\mathbf{V}}_\perp + \tilde{\mathbf{V}}_\parallel\right] = \mathcal{F}^{-1}\left[\tilde{\mathbf{V}}_\perp\right] + \mathcal{F}^{-1}\left[\tilde{\mathbf{V}}_\parallel\right] \\ &= \mathbf{V}_\perp + \mathbf{V}_\parallel \ . \end{aligned} \tag{58}$$