# Classical Electrodynamics

## Week 14

1. Consider a particle of mass $m$ and charge $e$ in a constant electromagnetic field $\mathbf{E} = E\mathbf{e}_y$, $\mathbf{B} = B\mathbf{e}_z$ with $B > E/c$.

   (a) Find the trajectory of the particle in the non-relativistic case.

   (b) Find the trajectory of the particle in the relativistic case using a boost to move to a reference frame where the electric field vanishes.

   *Solution*

   (a) **Non-relativistic case:** we start with Newton's law:

   $$m\ddot{\mathbf{x}} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1}$$

   that we decompose along the three axis:

   $$\begin{cases} m\ddot{x} = eB\dot{y} \\ m\ddot{y} = e(E - B\dot{x}) \\ m\ddot{z} = 0 \end{cases} \tag{2}$$

   The third equation is immediate to solve:

   $$z(t) = v_{0z}t + z_0 \tag{3}$$

   In order to solve the first two coupled differential equation (independent of $z$), we will use a trick: we define a complex variable $u = x + iy$ and take $(1) + i(2)$ in equation (2):

   $$m(\ddot{x} + i\ddot{y}) = eB\dot{y} + ie(E - B\dot{x}) = -ieB(\dot{x} + i\dot{y}) + ieE \tag{4}$$

   The equation obeyed by $u$ is:

   $$\ddot{u} + i\frac{eB}{m}\dot{u} = i\frac{eE}{m} \tag{5}$$

   This is a linear differential equation, the solution is a sum of a particular solution and the solutions of the homogeneous equation. So we have:

   $$\dot{u}(t) = \frac{E}{B} + Ce^{-i\omega t} \tag{6}$$

   where $\omega = \frac{eB}{m}$ and $C$ is a complex constant that we determine using the initial condition $\dot{u}(0) = v_{0x} + iv_{0y}$. We then read $\dot{x}(t)$ the real part of $\dot{u}(t)$ and $\dot{y}(t)$ its imaginary part:

   $$\begin{cases} \dot{x}(t) = \frac{E}{B} + \left(v_{0x} - \frac{E}{B}\right)\cos(\omega t) + v_{0y}\sin(\omega t) \\ \dot{y}(t) = \left(\frac{E}{B} - v_{0x}\right)\sin(\omega t) + v_{0y}\cos(\omega t) \end{cases} \tag{7}$$

We integrate once more and get the final result:

$$\begin{cases} x(t) = \frac{E}{B}t + \left(\frac{v_{0x}}{\omega} - \frac{E}{\omega B}\right)\sin(\omega t) + \frac{v_{0y}}{\omega}\left[1-\cos(\omega t)\right] + x_0 \\ y(t) = \left(\frac{E}{\omega B} - \frac{v_{0x}}{\omega}\right)\left[1-\cos(\omega t)\right] + \frac{v_{0y}}{\omega}\sin(\omega t) + y_0 \\ z(t) = v_{0z}t + z_0 \end{cases} \tag{8}$$

We can see that this trajectory is a circle with a moving center of coordinates $\mathbf{a} = \left(\frac{E}{B}t + \frac{v_{0y}}{\omega} + x_0, -\frac{v_{0x}}{\omega} + y_0, v_{0z}t + z_0\right)$. The radius is:

$$R^2 = ||\mathbf{x}(t) - \mathbf{a}||^2 = \left(\frac{v_{0x}}{\omega} - \frac{E}{\omega B}\right)^2 + \frac{v_{0y}^2}{\omega^2} \tag{9}$$

so the trajectory is a circular motion of radius $R$ around a center in a linear motion of speed $\mathbf{v} = \frac{E}{B}\mathbf{e}_x$.

(b) **Relativistic case:** We start by boosting to a frame $\mathcal{R}'$ where $\mathbf{E} = 0$. This is done in exercise 3: we need a boost $\vec{\beta} = \frac{1}{c}\frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{E}{B}\mathbf{e}_x$ and we are left with: $\mathbf{B}' = \sqrt{B^2 - \frac{E^2}{c^2}}\mathbf{e}_z$. We have a charged particle in a constant magnetic field!

In the generic relativistic case, we can use the equation:

$$\frac{dp^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu\,. \tag{10}$$

The time component gives the evolution of the energy:

$$\frac{d\mathcal{E}}{d\tau} = \frac{q}{c}\mathbf{E}'\cdot\mathbf{v} = 0 \quad\Rightarrow\quad \frac{d\mathcal{E}}{dt} = 0, \tag{11}$$

so the velocity $v = v_0$ and thus $\gamma = \gamma_0$ are constant. Then the space components give:

$$\frac{d\mathbf{p}}{d\tau} = m\gamma\frac{d\mathbf{v}}{d\tau} = \gamma q\mathbf{v}\times\mathbf{B}' \quad\Rightarrow\quad \frac{d\mathbf{p}}{dt} = m\gamma\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B}'\,, \tag{12}$$

which is the same equation as in the non-relativistic case with the replacement

$$\omega_B = \frac{qB'}{\gamma m} \tag{13}$$

where $\gamma$ is $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ with $v$ the velocity at $t = 0$.

Therefore, in the frame $\mathcal{R}'$, the trajectory of the particle is a helix: it is circular in the plane $(x', y')$ and linear along $z'$. In formulas, it is:

$$\begin{cases} x'(t') = \frac{v'_{0x}}{\omega_B}\sin(\omega_B t') + \frac{v'_{0y}}{\omega}\left[1-\cos(\omega_B t')\right] + x'_0 \\ y'(t') = -\frac{v'_{0x}}{\omega_B}\left[1-\cos(\omega_B t')\right] + \frac{v_{0y}}{\omega_B}\sin(\omega_B t') + y'_0 \\ z'(t') = v'_{0z}t' + z'_0 \end{cases} \tag{14}$$

where

$$\omega_B = \frac{qB'}{\gamma_0 m} = \frac{q}{m}\sqrt{B^2 - \frac{E^2}{c^2}}\sqrt{1 - \frac{(v'_0)^2}{c^2}} \tag{15}$$

We now need to boost back to the frame $\mathcal{R}$ with a boost $\vec{\beta} = -\frac{E}{B}\mathbf{e}_x$ to get the full solution. We will not do it explicitly but we will describe the trajectory: in $\mathcal{R}'$ it is a circle in the plane $(x', y')$ of fixed center and of radius $R' = \frac{\sqrt{(v'_{0x})^2+(v'_{0y})^2}}{\omega_B}$. In $\mathcal{R}$, the circle gets contracted along the $x$ direction but not along the $y$ direction: it becomes an ellipse of semi axis $a = R' = \frac{\sqrt{(v'_{0x})^2+(v'_{0y})^2}}{\omega_B}$ and $b = \frac{R'}{\gamma_E} = \frac{\sqrt{(v'_{0x})^2+(v'_{0y})^2}}{\omega_B}\sqrt{1-\frac{E^2}{B^2}}$.
The center of the ellipse is now moving with a constant velocity $\mathbf{v} = \frac{E}{B}\mathbf{e}_x$ like in the non-relativistic case.

2. *Relativistic Larmor formula*
Calculate the energy radiated per unit time $\frac{d\mathcal{E}}{dt}$ by a charged particle as a function of its velocity and the background electromagnetic fields. For simplicity, analyse the following cases:

   (a) $\mathbf{v} \parallel \mathbf{E}$

   (b) $\mathbf{v} \parallel \mathbf{B}$

   (c) $\mathbf{v} \parallel \mathbf{E} \times \mathbf{B}$

   (d) In the case of circular accelerator (synchrotron), with $\mathbf{E} = 0$, show that the energy loss is proportional to $\mathcal{E}^4$. **Hint:** Use the Larmor radius $\rho = \frac{m\gamma v}{qB}$ and the relativistic approximation $v \sim c$.

*Solution*

The relativistic Larmor formula is given by

$$\frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\epsilon_0 c^3} a^\mu a_\mu \,. \tag{16}$$

The charged particle follows the relativistic equation of motion

$$m a^\mu = \frac{q}{c} F^{\mu\nu} u_\nu \,. \tag{17}$$

With this we can express the power radiated as a function of the velocity

$$\frac{d\mathcal{E}}{dt} = \frac{q^2}{6\pi\epsilon_0 c^3}\frac{q^2}{m^2c^2} F^{\mu\alpha} F_{\mu\beta} u_\alpha u^\beta \,. \tag{18}$$

The Larmor formula is invariant under Lorentz transformations. This allows us to apply it in a convenient reference frame. We choose here the particle rest frame since there the velocity is simply given by

$$u'^\mu = (c, \mathbf{0}) \,. \tag{19}$$

And the Larmor formula becomes:

$$\frac{d\mathcal{E}}{dt} = \frac{1}{6\pi\epsilon_0 c}\frac{q^4}{m^2c^4} F'^{i0} F'_{i0} c^2 = \frac{q^4}{6\pi\epsilon_0 m^2 c^3} E'^2 \,. \tag{20}$$

All we need to do now is to compute the electric field in the particle rest frame $\mathcal{R}'$. To pass from $\mathcal{R}$ to $\mathcal{R}'$ we need to do a boost of speed $\mathbf{v}$ of the particle in $\mathcal{R}$. The transformation properties of the fields are given by

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \tag{21}$$
$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + v \times \mathbf{B}) \tag{22}$$

Let's consider the three proposed cases

(a) $\mathbf{v} \parallel \mathbf{E}$
$$E'^2 = E^2 + \gamma^2 v^2 B^2 \sin^2(\theta)\,, \tag{23}$$
where $\theta$ is the angle between $\mathbf{E}$ and $\mathbf{B}$. With $\mathbf{B} = 0$, this example corresponds to the case of a linear particle accelerator. It can be observed that in this case the energy losses are independent on the energy.

(b) $\mathbf{v} \parallel \mathbf{B}$
$$E'^2 = E^2(\cos^2(\theta) + \gamma^2 \sin^2(\theta))\,. \tag{24}$$

(c) $\mathbf{v} \parallel \mathbf{E} \times \mathbf{B}$
$$E'^2 = \gamma^2(\mathbf{E} + \mathbf{v} \times \mathbf{B})^2 = \gamma^2(E^2 + v^2B^2 - 2EvB\sin(\theta))\,. \tag{25}$$
With $\mathbf{E} = 0$, this example corresponds to the case of a circular particle accelerator (synchrotron).

(d) If we express the field $B$ as a function of the Larmor radius $\rho = \frac{m\gamma v}{qB}$ we obtain
$$\begin{aligned}\frac{d\mathcal{E}}{dt} &= \frac{q^4}{6\pi\epsilon_0 m^2 c^3}\gamma^2 B^2 v^2 = \frac{q^4}{6\pi\epsilon_0 m^2 c^3}\frac{m^2\gamma^4 v^4}{\rho^2} = \\ &\approx \frac{q^4}{6\pi\epsilon_0 m^2 c^3}\frac{\mathcal{E}^2\gamma^2}{\rho^2} = \frac{q^4}{6\pi\epsilon_0 m^4 c^7}\frac{\mathcal{E}^4}{\rho^2}\,,\end{aligned} \tag{26}$$
where in the second line we used $v \approx c$ and $\mathcal{E} = \gamma mc^2$. This shows that the energy losses, for fixed radius, are proportional to the fourth power of the energy.

3. *Kramers-Kronig relations for refractive index.*

In this exercise we will review a few steps in the derivation of Kramers Kronig relations, and derive the relation between real and imaginary parts of the refractive index $\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$ based on the analytic properties of electric permittivity $\varepsilon(\omega)$.

(a) As a first step, we want to study the behavior of $\varepsilon(\omega)$ for real and very large frequencies. You can derive that such behavior is universal, regardless of whether the material is a dielectric or a conductor.
Compute the polarization $\vec{P}$ of the body when the frequency of the incident (monochromatic) electromagnetic wave is much larger than the motion frequency of any atomic electron, and derive the correspondent electric permittivity in this regime.
**Hint:** Motivate why at very high frequencies electric field can be thought of being uniform in space when determining electrons acceleration from the field.

(b) Argue that for $\omega \to \infty$ in any direction in the upper-half plane $\varepsilon(\omega) \to \epsilon_0$.

(c) Show that $\varepsilon(\omega)$ cannot acquire any real value on the upper half plane, except on the imaginary axes, where it is monotonically decreasing. From here, conclude that $\varepsilon(\omega) \neq 0$ in the upper-half $\omega$-plane.
**Hint 1:** It can be heplful to recall the theorem for any function $g$ of complex variables $z$, which states that the integral

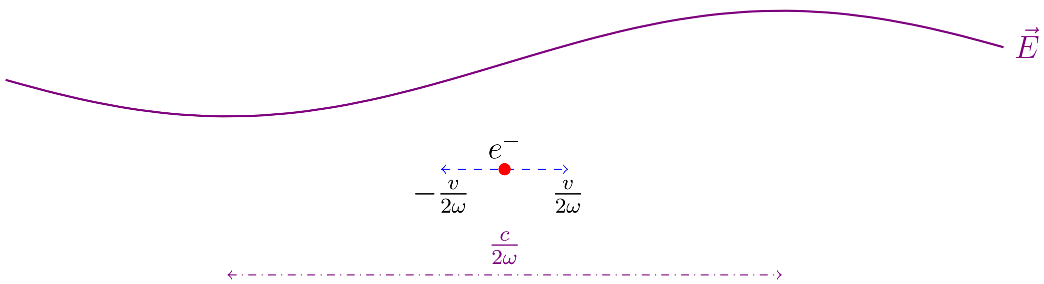
$$\frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{\mathrm{d}g(z)}{\mathrm{d}z} \frac{\mathrm{d}z}{g(z) - a} = \#_{\mathrm{zeros}} - \#_{\mathrm{poles}}, \tag{27}$$

where $a$ is any constant, $\#_{\mathrm{zeros}}$ is the number of zeros of the function $g(z)-a$ within the contour $\mathcal{C}$, and $\#_{\mathrm{poles}}$ is the number of poles of $g(z)-a$ inside $\mathcal{C}$.
**Hint 2:** The function $\varepsilon(\omega)$ always has a non-vanishing imaginary part when $\omega$ is real and different from 0. This is due to the dissipation any electromagnetic wave is subjected to when traveling through a medium.

(d) Assuming constant magnetic permeability, so that $\tilde{n}(\omega) = \sqrt{\varepsilon(\omega)\mu}$, derive the Kramers-Kronig relations for $\tilde{n}(\omega)$. Think about a good strategy for measuring the refractive index, $\tilde{n}(\omega)$, given the derived relations.

*Solution*

(a) For very high frequencies $\omega$ of the incident electromagnetic wave, we can regard the electrons as free, and neglect their interaction with one another and with the atomic nuclei. Their velocity $v$ is small compared to the speed of light $v \ll c$, therefore the distance $\frac{v}{\omega}$ travelled during one period of the electromagnetic wave is small compared to the wavelength $\frac{c}{\omega}$. For this reason, we can assume the wave to be uniform in space when de-




termining the velocity acquired by the electron in this field. The equation of motion is:

$$m_e \frac{\mathrm{d}\vec{v'}}{\mathrm{d}t} = e\vec{E} = e\vec{E}_0\, \mathrm{e}^{i\vec{k}\cdot x - i\omega t} \simeq e\vec{E}_0\, \mathrm{e}^{-i\omega t} \tag{28}$$

where $m_e$ is the electron mass and $\vec{v'}$ is the additional velocity acquired by the electron. The solution is $\vec{v'} = \frac{ie}{m_e\omega}\vec{E}$. The electron displacement due to the field is $\dot{\vec{r}} = \vec{v'}$, hence $\vec{r} = -\frac{e\vec{E}}{m_e\omega^2}$. We can finally compute the polarization $\vec{P}$ – which is the dipole moment per unit volume – by summing over the displacement of the $N$ electrons in a unit volume of the medium

$$\vec{P} = \sum_{i=1}^{N} e\vec{r}_i = -\frac{e^2 N}{m_e\omega^2}\vec{E}. \tag{29}$$

By definition of electric induction, we can finally derive $\varepsilon$:

$$\vec{D}(\omega) = \varepsilon(\omega)\vec{E}(\omega) = \varepsilon_0\vec{E}(\omega) + \vec{P}(\omega) \implies \varepsilon(\omega \gg 1) = \varepsilon_0\left(1 - \frac{Ne^2}{m_e\omega^2}\right). \tag{30}$$

(b) Electric permittivity in the context of non-linear optics has been defined as:

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \int_0^\infty \mathrm{e}^{i\omega\tau} f(\tau)\mathrm{d}\tau\right), \tag{31}$$

with $f(\tau)$ being the response function depending on time and on properties of the medium, and relates $\vec{D}$ to $\vec{E}$.
If we send $\omega \to \infty$ in such a way that $\mathfrak{Im}(\omega) \to +\infty$ then the integrand in (31) is suppressed by $\mathrm{e}^{-\mathfrak{Im}(\omega)\tau}$, so $\varepsilon \to 1$.
Instead, if we send $\omega \to \infty$ along a path which keeps $\mathfrak{Im}(\omega)$ finite, then $\mathfrak{Re}(\omega) \to 0$, and so the integrand in (31), which contains $\mathrm{e}^{i\mathfrak{Re}(\omega)\tau}$, is strongly oscillating and the integral goes to zero.

(c) We know from the reality of the fields $\vec{E}$ and $\vec{D}$ that the response function $f(\tau)$ in (31) is real. This implies that $\varepsilon(-\omega^*) = \varepsilon(\omega)^*$. It means that permittivity is real on the imaginary axis, $\varepsilon(i\mathfrak{Im}(\omega)) \in \mathbb{R}$.

Let's now define the function $g(\omega) = \varepsilon(\omega)/\varepsilon_0 - 1$, which also has to be real on the imaginary axis. Let's call $g_0 = g(\omega \to i0)$, $g_0 \in \mathbb{R}$, and let's assume $g_0$ is finite, the study of the infinite $g_0$ case is a slight modification of our following reasoning.

Let's apply Hint 1 to the function $g(\omega)$, and choose $a$ to be a real number. Let's also choose as integration contour $\mathcal{C}$ the real line and an infinite semi–circle in the upper half plane, as shown on the left side of figure 1. Since causality implies $\varepsilon(\omega)$ doesn't have any pole in the upper half-plane, the same applies to $g(\omega) - a$, so the integral in (27) evaluates to the number of zeros of $g(\omega) - a$, i. e. the number of points for which $g(\omega) = a$.

We now need to compute the integral in (27); in order to do so, we can rewrite it in this form

$$(27) = \oint_{\mathcal{C}'} \frac{\mathrm{d}g}{g - a}, \tag{32}$$

where the integration is carried out around a contour $\mathcal{C}'$ in the plane of the complex variable $g$, and it is the image through the map $g(\omega)$ of the contour $\mathcal{C}$ in the original $\omega$ plane.

In particular, through this map, the all infinite semi-circle is mapped on to the point $g = 0$ (since we have proved in point 2 that $g(\omega \to \infty) = 0$), while the origin $\omega = 0$ is mapped on to another real point $g_0$. The right and left halves of the real axis of $\omega$ are mapped on to some very complicated (and generally self-intersecting) curves; such curves are entirely contained in the upper or lower halves of the $g$-plane respectively: it is important to remark that these curves nowhere meet the real axis (except at $g = 0$ and $g_0$), since $g(\omega)$ is never real for $\omega \in \mathbb{R} \setminus \{0\}$, as per Hint 2. Because of these properties of the contour $\mathcal{C}'$, the integral in (32) evaluates to 1 if $0 < a < g_0$ (as depicted in figure 1 on the right hand side), or to 0 if $a > g_0$ or $a < 0$. Therefore, we conclude that $g(\omega)$ takes, in the

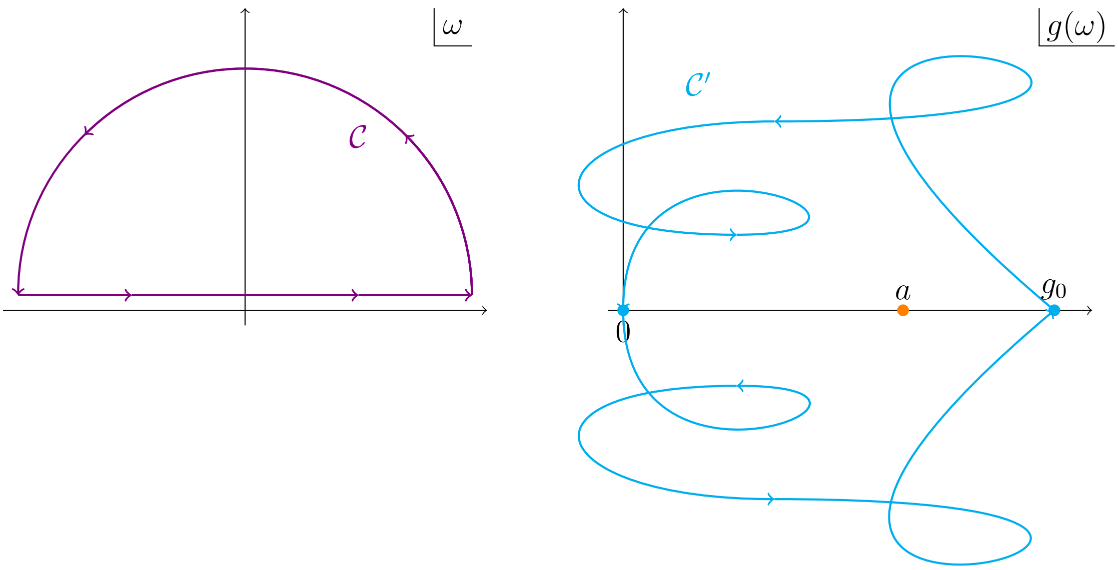



Figure 1: Integration contours in the $\omega$ and $g(\omega)$ complex planes.

upper-half $\omega$-plane, each value of $a$ in the range $0 < a < a_0$ exactly once, and never takes real values at all outside of this range.
The result of this computation implies that $g(\omega)$ must be monotonically decreasing between $g_0$ and 0 along the imaginary axis, otherwise there would be real values taken twice. Following from this, all the real values between 0 and $g_0$ are taken along the imaginary $\omega-$axis, so there can be no other point in the $\omega$-plane for which $g$, and so $\varepsilon$, is real.
Finally, since $\varepsilon(\omega)$ is nowhere real outside of the imaginary axis, in particular it can never be 0. On the real axis it is monotonically decreasing from $\varepsilon_0(g_0 + 1)$ to $\varepsilon_0$, so it never evaluates to 0 either.

(d) In the previous point we proved that $\varepsilon(\omega) \neq 0$ in the upper half-plane. Recall also that causality implies that $\varepsilon(\omega)$ is analytic in the upper half-plane. From these two facts, together with the relation $\tilde{n}(\omega) = \sqrt{\varepsilon(\omega)\mu}$, it follows that $\tilde{n}(\omega)$ has no branch points and it has the same analytic properties as $\varepsilon(\omega)$. Now we can immediately apply the Kramers-Kronig relations for the real and imaginary part of $\tilde{n}(\omega)$, since this function satisfys all the necessary analytic properties:

$$n(\omega) = n_0 + \frac{2}{\pi}\mathcal{P}\int_0^{+\infty} \frac{\omega'\kappa(\omega')}{\omega'^2 - \omega^2} d\omega', \tag{33}$$

where $n_0 = \sqrt{\varepsilon_0\mu}$.
Let us now turn to the question about experimental measurement of refractive index. The electromagnetic wave inside the medium decays due

to the imaginary part of refractive index:

$$E = E_0 e^{-i(\omega t - kx)} = E_0 e^{-i(\omega t - \tilde{n}\frac{\omega}{c}x)} = E_0 e^{-i(\omega t - n\frac{\omega}{c}x)} e^{-\kappa\frac{\omega}{c}x} \tag{34}$$

Then the intensity of the light going through the material decays exponentially as

$$I \sim |E|^2 = E_0^2 e^{-\frac{2\kappa\omega}{c}x} \Rightarrow I = I_0 e^{-\alpha x}, \tag{35}$$

where

$$\alpha(\omega) = \frac{2\kappa(\omega)\omega}{c}. \tag{36}$$

This is called an absorption coefficient and it can be easily measured by measuring the intensity. Replacing $\kappa(\omega)$ with $\alpha(\omega)$ in (33) we get the formula for the real part of the refractive index:

$$n(\omega) = n_0 + \frac{c}{\pi}\mathcal{P}\int_0^{+\infty} \frac{\alpha(\omega')}{\omega'^2 - \omega^2} d\omega' \ . \tag{37}$$

By measuring a profile of the function $\alpha(\omega)$ for multiple frequencies $\omega$ one can then perform the numerical integration to get the values of $n(\omega)$. In real physical systems one doesn't need to measure $\alpha(\omega)$ along the whole real line. Instead, the absorption coefficient decays to zero outside of some physical window of frequencies. That is, some finite number of measurements is enough to also recover the behavior of the real part of the refractive index.

# Bonus Question

1. *Lorentz transformation of electromagnetic fields III*
   Consider a constant electromagnetic field $\{\mathbf{E}, \mathbf{B}\}$ such that $\mathbf{E} \cdot \mathbf{B} = 0$ in the reference frame $\mathcal{R}$.

   (a) Find a reference frame $\mathcal{R}'$ where either $\mathbf{E}' = 0$ or $\mathbf{B}' = 0$.

   (b) Is this always possible? Is the solution unique?

   (c) Compute the magnitudes of $\mathbf{E}'$ and $\mathbf{B}'$ in the reference frame $\mathcal{R}'$.

   *Solution*

   (a) If $\mathbf{E}$ and $\mathbf{B}$ are orthogonal in one reference frame they will be orthogonal in any other reference frame. We need to do a boost in the direction orthogonal to the plane of $\mathbf{E}$ and $\mathbf{B}$. Indeed, if we want, for example, to cancel $\mathbf{E}$, we need a boost $\vec{\beta}$ that is orthogonal to $\mathbf{E}$, so that $\mathbf{E}'_\parallel = \mathbf{E}_\parallel = 0$, but also that is not parallel to $\mathbf{B}$, otherwise we end up with

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + c\vec{\beta} \times \mathbf{B}\right) = \gamma\mathbf{E}_\perp \, , \tag{38}$$

   that is different from zero by construction. Suppose that we want to have $\mathbf{E}' = 0$, we need

$$\mathbf{E}_\perp + c\boldsymbol{\beta} \times \mathbf{B} = \mathbf{E} + c\boldsymbol{\beta} \times \mathbf{B} = 0 \tag{39}$$

Concerning the length of $\boldsymbol{\beta}$, we see that it must be

$$\beta = \frac{E}{cB}, \tag{40}$$

and this relation tells us that the boost is possible only if $E < cB$.
The direction of $\boldsymbol{\beta}$ can be found with the rule of the right hand, and it is

$$\frac{\boldsymbol{\beta}}{\beta} = \frac{\mathbf{E}}{E} \times \frac{\mathbf{B}}{B}. \tag{41}$$

Therefore, the solution is

$$\boldsymbol{\beta} = \frac{1}{c}\frac{\mathbf{E} \times \mathbf{B}}{B^2}. \tag{42}$$

If instead we want to cancel $\mathbf{B}$, the boost that we need to do is still in the direction orthogonal to both $\mathbf{E}$ and $\mathbf{B}$. The condition for $\mathbf{B}'_\perp = 0$ is

$$\mathbf{B} - \frac{1}{c}\boldsymbol{\beta} \times \mathbf{E} = 0, \tag{43}$$

and the solution is

$$\boldsymbol{\beta} = c\frac{\mathbf{E} \times \mathbf{B}}{E^2}, \tag{44}$$

so that the length of $\boldsymbol{\beta}$ is

$$\beta = \frac{cB}{E}, \tag{45}$$

meaning that the boost is possible only if $E > cB$.

(b) The solution is not unique because, once we have found a reference frame in which $\mathbf{E}'$ is zero, it will remain zero after an arbitrary boost in the direction of $\mathbf{B}'$, and an analogous argument can be made if instead $\mathbf{B}'$ is zero: it will still be zero after an arbitrary boost in the direction of $\mathbf{E}'$. However, if $E = cB$, it is impossible to cancel any of the two fields. This should not be surprising, since that is the case of the electromagnetic waves, and we know that it is impossible to find a reference frame in which a wave is canceled.

(c) Using the Lorentz invariant $E^2 - c^2B^2$, we have

$$E' = \sqrt{E^2 - c^2B^2}, \tag{46}$$

if we cancel $\mathbf{B}$ and

$$B' = \sqrt{B^2 - \frac{E^2}{c^2}}, \tag{47}$$

if we cancel $\mathbf{E}$.