Classical Electrodynamics

Week 13

1. *Relativistic Doppler effect*
An electromagnetic wave is described by the fields:

$$\mathbf{E} = \mathbf{E}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x}) \; , \qquad \mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E} \; , \qquad \mathbf{E}_0\cdot\mathbf{k} = 0 \; , \tag{1}$$

where ω is the frequency in the laboratory reference frame and $\mathbf{k} = \omega/c\,\mathbf{e}_z$ is wave vector. An observer moves at constant velocity v along the z-axis. Find the frequency ω' of the wave in the reference frame of the moving observer. Show that $\left(\frac{\omega}{c}, \mathbf{k}\right)$ is a four-vector.
Hint: The phase of an electromagnetic wave is Lorentz invariant.

Solution

The phase of a wave is Lorentz invariant because we can identify it with the number of wave peaks in a certain time interval. Indeed, if observer $\mathcal{R}$ counts n peaks in a certain interval Δt, observer $\mathcal{R}'$ will declare the interval longer, and the peaks more rare, but will still acknowledge that n peaks reached observer $\mathcal{R}$ between the beginning and the end of the experiment. You can also check directly the invariance of the phase: $F_{\mu\nu}$ is proportional to $\sin(\omega t - \mathbf{k}\cdot\mathbf{x})$, and since it transforms as a tensor under Lorentz transformations, also $F'_{\mu\nu}$ will be. Therefore we just have to write the argument of the sine in terms of the transformed coordinates z' and t':

$$ct' = \gamma ct - \beta\gamma z \tag{2}$$
$$z' = -\beta\gamma ct + \gamma z, \tag{3}$$

and so

$$ct = \gamma ct' + \beta\gamma z' \tag{4}$$
$$z = \beta\gamma ct' + \gamma z'. \tag{5}$$

The phase becomes:

$$\omega t - kz = \gamma(\omega - kv)t' - \gamma\left(k - \omega\frac{v}{c^2}\right)z'. \tag{6}$$

Now, the observer $\mathcal{R}'$ measures the frequency by counting the wave peaks that reach him, who is sitting at constant z' in a certain time interval, therefore

$$\omega' = \gamma(\omega - kv) = \gamma\omega(1-\beta) = \frac{\omega}{\gamma(1+\beta)}, \tag{7}$$

where we have used $\omega = ck$. Notice that this is not just the effect of time dilatation: the frequency is defined with respect to two different spacetime intervals by the two observers (see spacetime diagram in fig. 1 and the comment at the end of the solution). The frequency itself is $\nu = \omega/(2\pi)$, so

$$\nu' = \nu\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}. \tag{8}$$

If the observer $\mathcal{R}'$ moves in the same direction of the wave - that is, away from the source, with $v > 0$, we see that the frequency lowers, that is, the wave is red-shifted.

The fact that $k^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right)$ transforms as a four-vector immediately follows from eq. (6), after defining ω' and k' such that $\omega t - kz = \omega' t' - k' z'$. That same conclusion could be reached by noticing that the phase can be written as follows:

$$\omega t - \mathbf{k} \cdot \mathbf{x} = k^\mu g_{\mu\nu} x^\nu, \tag{9}$$

with $g_{\mu\nu}$ the Minkowski metric. Invariance of the phase can be written as

$$k'^\mu g_{\mu\nu} x'^\nu = k^\mu g_{\mu\nu} x^\nu, \tag{10}$$

which of course implies that x^μ and k^μ transform as four-vectors.

Alternative method The following is equivalent to observing that the phase is Lorentz invariant, but might be more intuitive. The phase diagram 1 shows in dotted lines the lines of constant phase, which travel at the speed of light. The time interval between one peak and the next is cT=length(AB) in the frame $\mathcal{R}$ and cT'=length(AC) in the frame $\mathcal{R}'$. The frequencies are the inverse of this time interval, so we just need to compute T' as a function of T. The point C is the crossing point of the following two world-lines:

$$\text{light:} \quad ct = cT + z \tag{11}$$

$$\text{observer:} \quad ct = \frac{c}{v} z. \tag{12}$$

Therefore:

$$cT + z_C = \frac{c}{v} z_C, \tag{13}$$

that implies

$$z_C = \frac{vT}{1 - \frac{v}{c}} \tag{14}$$

$$t_C = \frac{z_C}{v} = \frac{T}{1 - \frac{v}{c}}. \tag{15}$$

We are ready to compute the length of the segment AC, and we must do it with the Minkowski metric:

$$cT' = \text{length(AC)} = \sqrt{(ct_C)^2 - z_C^2} = \frac{\sqrt{c^2 - v^2}}{1 - \frac{v}{c}} T = cT \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}. \tag{16}$$

Since $\nu = 1/T$, we recover the previous result. Finally, let us stress again that, as evident from the formula, the Doppler effect is not the result of time dilatation on the interval T: indeed the interval would even contract, if the observer $\mathcal{R}'$ had been running towards the source ($v < 0$). Remember, instead, that the time interval whose length is γT is the segment AD in the spacetime diagram, as it is easy to verify.

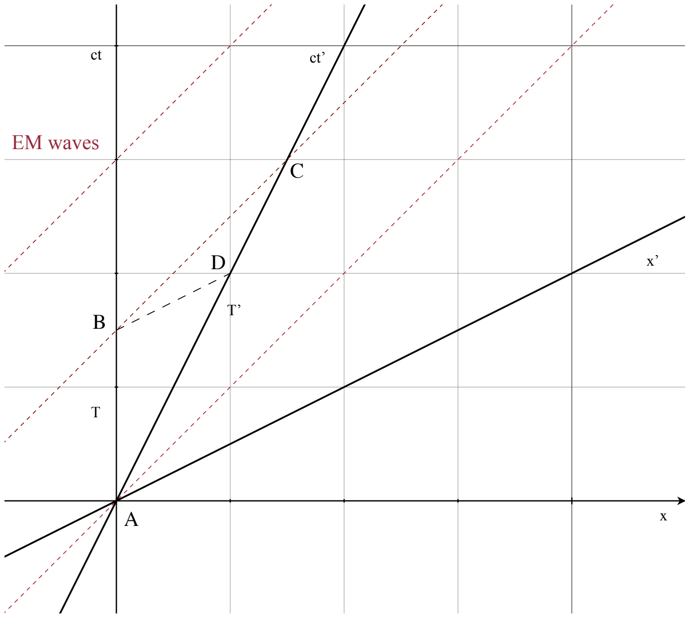

Figure 1: Spacetime diagram of the situation. The dotted lines represent the trajectory of points of constant phase of the electromagnetic wave.

2. A conducting loop with a rectangular shape of sides a' and b' supports the current I' (in the reference frame where the loop is at rest). The cross section of the wire of the loop is S'. The loop moves at constant speed v in the direction parallel to the side of length a'.
Find the charge distribution and currents in each side of the loop in the reference frame of the laboratory. Comment on your results.

Solution

We know that the object transforming as a Lorentz four-vector is the current density $j^\mu = (c\rho, \mathbf{j})$. So we will compute j'^μ in the reference frame of the loop $\mathcal{R}'$ and transform it to the laboratory frame $\mathcal{R}$. The relation between the currents in the two frames is:

$$j'^\mu(x') = \Lambda^\mu{}_\nu j^\nu(\Lambda^{-1}x') = \Lambda^\mu{}_\nu j^\nu(x) \tag{17}$$

which implies that

$$j^\mu(x) = (\Lambda^{-1})^\mu{}_\nu j'^\nu(x') \tag{18}$$

where Λ and Λ^{-1} are the matrices:

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{19}$$

(The frame attached to the loop $\mathcal{R}'$ moves at velocity $v\mathbf{e}_x$ in the laboratory frame $\mathcal{R}$ and we define $\beta = v/c$. Λ is the transformation that takes us from $\mathcal{R}$ to $\mathcal{R}'$ i.e $x' = \Lambda x$).

In side 1 of the loop (the side where I is in the positive x' direction), in the loop frame : there is no charge density $\rho'_1 = 0$ and the current density is $\mathbf{j}'_1 = I'/S'\mathbf{e}_x$

in the volume of the wire. In the laboratory frame, we have:

$$\begin{cases} c\rho_1 = \gamma(c\rho_1' + \beta j_1') = \gamma\beta\frac{I'}{S'} \\ j_1 = \gamma(\beta c\rho_1' + j_1') = \gamma\frac{I'}{S'} \end{cases} \tag{20}$$

Now the wire of side 1 is positively charged and its total charge is $aS_1\rho_1$, with a and S_1 the length and section of the wire measured in $\mathcal{R}$ taking into account Lorentz contraction, which is: $a = a'/\gamma$, $S_1 = S'$. So the total charge is:

$$Q_1 = \frac{a'}{\gamma}S'\frac{1}{c}\gamma\beta\frac{I'}{S'} = \frac{v}{c^2}a'I'. \tag{21}$$

The current in side 1 is given by:

$$I_1 = j_1 S_1 = \gamma I'. \tag{22}$$

Notice that while the wire is better described by a linear charge density λ and current I, the quantities that transform well under Lorentz boosts are the *volumic* charge density ρ and current density $\mathbf{j}$. In order to pass from one description to the other, we have to introduce the section of the wire.

In side 2 of the loop (following the flow of the current), $\rho_2' = 0$ and $\mathbf{j}_2' = I'/S'\mathbf{e}_y$. Applying Lorentz transformation:

$$\begin{cases} c\rho_2 = \gamma c\rho_2' = 0 \\ j_2 = j_2' = \frac{I'}{S'}, \end{cases} \tag{23}$$

but now the section of the wire gets Lorentz contracted. So $S_2 = S'/\gamma$ and:

$$\begin{cases} Q_2 = 0 \\ I_2 = j_2' S_2 = \frac{I'}{\gamma}. \end{cases} \tag{24}$$

We can replace I' by $-I'$ to get the charge and current in sides 3 and 4:

$$\begin{cases} Q_3 = -\frac{v}{c^2}a'I' \\ I_3 = -\gamma I', \end{cases} \tag{25}$$

and

$$\begin{cases} Q_4 = 0 \\ I_4 = -\frac{I'}{\gamma}. \end{cases} \tag{26}$$

So in the laboratory frame a positive charge appears in the side where the current flows in the direction of the movement of the loop and a negative charge appears when the current flows in the opposite direction, the total charge of the system staying zero. The current is not the same in the four sides : it is bigger in the sides parallel to the movement and smaller in the sides perpendicular to the movement.

We can interpret the result as a “real” current I'/γ flowing in the rectangle which corresponds to the original current where charges are slowed down by time dilatation. The current in sides 1 and 3 is bigger (in absolute value) because of the charge density moving at velocity v which creates additionnal current. One can check that:

$$I_1 = \frac{I'}{\gamma} + \gamma I' \left(1 - \frac{1}{\gamma^2}\right) = \frac{I'}{\gamma} + \gamma\beta^2 I' = \frac{I'}{\gamma} + \rho_1 S_1 v, \tag{27}$$

which shows the current I_1 as a sum of the current flowing though all the rectangle (first term) and the current created by the charges (second term).

An analytical solution This method has the advantage of not introducing the section S' of the wire. The numbering of the sides of the loop is the same as in the previous solution. In the reference frame of the loop $\mathcal{R}'$ we write the charge and current density:

$$\rho'(x', y', z') = 0 \tag{28}$$

and

$$\begin{aligned}
\mathbf{j}'(x', y', z') =& I'\delta(y')\delta(z')\mathbf{e}_x \left[\Theta(x') - \Theta(x' - a')\right] \qquad \text{side 1} \\
&+ I'\delta(x' - a')\delta(z')\mathbf{e}_y \left[\Theta(y') - \Theta(y' - b')\right] \qquad \text{side 2} \\
&- I'\delta(y' - b')\delta(z')\mathbf{e}_x \left[\Theta(x') - \Theta(x' - a')\right] \qquad \text{side 3} \\
&- I'\delta(x')\delta(z')\mathbf{e}_y \left[\Theta(y') - \Theta(y' - b')\right] \qquad \text{side 4}
\end{aligned} \tag{29}$$

where the delta functions express that the wire is one-dimensional and the theta function are here to ensure that the current is only non zero in the loop. We then transform the four vector j'^{μ} to the laboratory frame $\mathcal{R}$:

$$\begin{pmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \beta\gamma j'_x \\ \gamma j'_x \\ j'_y \\ j'_z \end{pmatrix} = \begin{pmatrix} \beta\gamma I' \left[\delta(y')\delta(z') - \delta(y' - b')\delta(z')\right] \left[\Theta(x') - \Theta(x' - a')\right] \\ \gamma I' \left[\delta(y')\delta(z') - \delta(y' - b')\delta(z')\right] \left[\Theta(x') - \Theta(x' - a')\right] \\ I' \left[\delta(x' - a')\delta(z') - \delta(x')\delta(z')\right] \left[\Theta(y') - \Theta(y' - b')\right] \\ 0 \end{pmatrix}$$

The coordinates in $\mathcal{R}'$ are related to coordinates in $\mathcal{R}$ by:

$$\begin{cases} ct' = \gamma ct - \beta\gamma x \\ x' = -\beta\gamma ct + \gamma x \\ y' = y \quad \text{and} \quad z' = z \end{cases} \tag{30}$$

so:

$$j^{\mu} = \begin{pmatrix} \beta\gamma I' \left[\delta(y)\delta(z) - \delta(y - b')\delta(z)\right] \left[\Theta(-\beta\gamma ct + \gamma x) - \Theta(-\beta\gamma ct + \gamma x - a')\right] \\ \gamma I' \left[\delta(y)\delta(z) - \delta(y - b')\delta(z)\right] \left[\Theta(-\beta\gamma ct + \gamma x) - \Theta(-\beta\gamma ct + \gamma x - a')\right] \\ I' \left[\delta(-\beta\gamma ct + \gamma x - a')\delta(z) - \delta(-\beta\gamma ct + \gamma x)\delta(z)\right] \left[\Theta(y) - \Theta(y - b')\right] \\ 0 \end{pmatrix}$$

We can simplify the Θ and δ functions as follows:

$$\begin{cases} \Theta(-\beta\gamma ct + \gamma x) - \Theta(-\beta\gamma ct + \gamma x - a') = \Theta(x - vt) - \Theta(x - vt - a'/\gamma) \\ \delta(-\beta\gamma ct + \gamma x) = \frac{1}{\gamma}\delta(x - vt) \\ \delta(-\beta\gamma ct + \gamma x - a') = \frac{1}{\gamma}\delta(x - vt - a'/\gamma) \end{cases} \tag{31}$$

Then we have:

$$\begin{aligned}\rho(x,y,z) =& \frac{\beta\gamma I'}{c}\delta(y)\delta(z)\left[\Theta(x-vt)-\Theta(x-vt-a'/\gamma)\right]\\ &-\frac{\beta\gamma I'}{c}\delta(y-b')\delta(z)\left[\Theta(x-vt)-\Theta(x-vt-a'/\gamma)\right]\end{aligned} \tag{32}$$

corresponding to the positive charge in side 1 and negative charge in side 3. The current density is:

$$\begin{aligned}\mathbf{j}(x,y,z) =& \gamma I'\delta(y)\delta(z)\left[\Theta(x-vt)-\Theta(x-vt-a'/\gamma)\right]\mathbf{e}_x \qquad \text{side 1}\\ &-\gamma I'\delta(y-b')\delta(z)\left[\Theta(x-vt)-\Theta(x-vt-a'/\gamma)\right]\mathbf{e}_x \qquad \text{side 3}\\ &+\frac{I'}{\gamma}\delta(x-vt)\delta(z)\left[\Theta(y)-\Theta(y-b')\right]\mathbf{e}_y \qquad \text{side 2}\\ &-\frac{I'}{\gamma}\delta(x-vt-a'/\gamma)\delta(z)\left[\Theta(y)-\Theta(y-b')\right]\mathbf{e}_y \qquad \text{side 4}\end{aligned} \tag{33}$$

The $x-vt$ dependence of the Θ functions indicate that the loop is moving in the x direction with speed v. Also the Lorentz contraction of the section of the wire appear as a $1/\gamma$ factor in the delta function. Everything combines to give the same result as in the first solution.

3. *The Breakthrough Starshot program*
The stress tensor associated to electromagnetic fields is

$$T^{\mu\nu} = \frac{1}{\mu_0 c^2}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right),$$

where $F^{\mu\nu}$ is the field-strength tensor.

Remark: The questions are formulated using the $(-,+++)$ convention for the Minkowski metric.

(a) Show that, in terms of the electromagnetic fields, the components of $T^{\mu\nu}$ reduce to:

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & \vec{S}/c \\ \vec{S}/c & -\tau_{ij}\end{pmatrix},$$

where $\mathcal{E}$ is the electromagnetic energy density, $\vec{S}$ is the Poynting vector and τ_{ij} are the components of the Maxwell stress tensor, defined as:

$$\tau_{ij} := \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right).$$

(b) Compute the divergence $\partial_\nu T^{\mu\nu} = -f^\mu$ and show that f^μ is a 4−vector. What physical quantities do its components represent? **Hint:** use Maxwell equations in presence of charges.

The Breakthrough Starshot program aims at developing an ultra-fast spaceship for interstellar missions. The prototype is made of a very light nanocraft attached to a lightsail, propelled by a ground-based light beamer. You can find a sketch of it in picture 2.

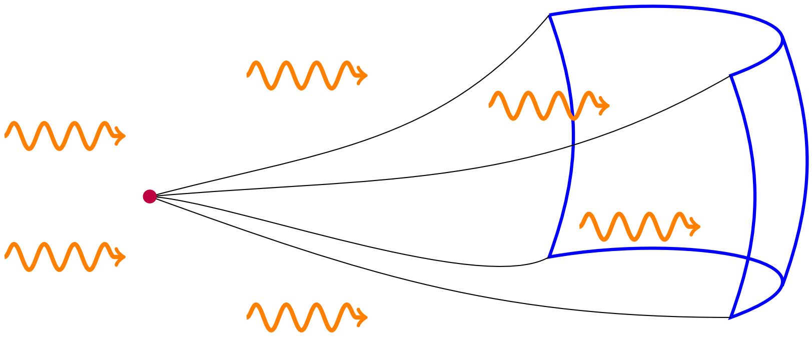

Figure 2: Sketch of a prototype of the Breakthrough Starshot program. In blue is depicted the lightsail, the electromagnetic wave arrives perpendicularly, to which is attached the purple nanocraft.

(c) Assuming the incident electromagnetic wave is perfectly reflected by the light sail (and the light sail is flat), compute the power per unit area w transferred to the light sail by an incident electromagnetic wave of amplitude $|\vec{E}| = c|\vec{B}|$ (in the reference frame where the light sail is at rest).

(d) The project aims at pushing the nanocraft to travel at speed $c/5$. What is the ratio between the power w' which needs to be injected in the light beam from Earth, and the power w transferred to the light sail in its reference frame?

Solution

(a) First of all, let's compute the contraction $F_{\alpha\beta}F^{\alpha\beta}$.

$$\begin{aligned} F_{\alpha\beta}F^{\alpha\beta} &= F^{0i}F_{0i} + F^{i0}F_{i0} + F^{ij}F_{ij} = \\ &= 2F^{0i}F_{0i} + F^{ij}F_{ij} \\ &= 2(c^2\vec{B}^2 - \vec{E}^2) \end{aligned} \tag{34}$$

where we used $F_{0i} = -E_i$ and $F_{ij} = c\epsilon_{ijk}B_k$. Then we have (remember here $\eta^{00} = -1$)

$$\begin{aligned} T^{00} &= \epsilon_0 F^{0\alpha}F^{0}{}_{\alpha} + \frac{\epsilon_0}{2}(c^2\vec{B}^2 - \vec{E}^2) \\ &= \epsilon_0 F^{00}F^{0}{}_{0} + \epsilon_0 F^{0i}F^{0}{}_{i} + \frac{\epsilon_0}{2}(c^2\vec{B}^2 - \vec{E}^2) \\ &= \epsilon_0\vec{E}^2 + \frac{\epsilon_0}{2}(c^2\vec{B}^2 - \vec{E}^2) \\ &= \frac{\epsilon_0}{2}(c^2\vec{B}^2 + \vec{E}^2) \equiv \mathcal{E}. \end{aligned} \tag{35}$$

Similarly,

$$T^{0i} = \frac{1}{\mu_0 c^2}F^{0\alpha}F^{i}{}_{\alpha} = \frac{1}{\mu_0 c^2}F^{0j}F_{ij} = \frac{1}{\mu_0 c}\epsilon_{ijk}E_jB_k \equiv \frac{S^i}{c}. \tag{36}$$

Finally,

$$
\begin{aligned}
T^{ij} &= \epsilon_0 \left(F^{i\alpha} F^{j}_{\ \alpha} + \frac{1}{2}((c^2 \vec{B}^2 - \vec{E}^2)\delta_{ij} \right) \\
&= \epsilon_0 \left(F^{i0} F^{j}_{\ 0} + F^{ik} F^{j}_{\ k} + \frac{1}{2}(\vec{E}^2 - c^2 \vec{B}^2)\delta_{ij} \right) \\
&= \epsilon_0 \left(-E_i E_j + \frac{1}{2}\delta_{ij} \vec{E}^2 \right) + \frac{1}{\mu_0} \left(\epsilon_{ikl} B_l \epsilon_{jkm} B_m - \frac{1}{2}\delta_{ij} \vec{B}^2 \right) \\
&= \epsilon_0 \left(-E_i E_j + \frac{1}{2}\delta_{ij} \vec{E}^2 \right) + \frac{1}{\mu_0} \left((\delta_{ij}\delta_{lm} - \delta_{im}\delta_{jl}) B_l B_m - \frac{1}{2}\delta_{ij} \vec{B}^2 \right) \\
&= \epsilon_0 \left(-E_i E_j + \frac{1}{2}\delta_{ij} \vec{E}^2 \right) + \frac{1}{\mu_0} \left(-B_i B_j + \frac{1}{2}\delta_{ij} \vec{B}^2 \right) \equiv -\tau_{ij} \, .
\end{aligned}
\tag{37}
$$

(b) We compute here the divergence of the stress tensor $\partial_\nu T^{\mu\nu}$.
Method 1:
For the time-component we have $\partial_\nu T^{0\nu} = \partial_0 T^{00} + \partial_i T^{0i} = \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + \partial_i T^{0i}$. We compute then

$$
\begin{aligned}
\partial_i T^{0i} &= \frac{1}{c} \partial_i S^i \\
&= \frac{1}{c\mu_0} \partial_i (\epsilon_{ijk} E_j B_k) \\
&= \frac{1}{c\mu_0} \vec{B} \cdot \underbrace{\left(\nabla \wedge \vec{E} \right)}_{-\partial_t \vec{B}} - \frac{1}{c\mu_0} \vec{E} \cdot \underbrace{\left(\nabla \wedge \vec{B} \right)}_{\mu_0 \vec{J} + \mu_0 \epsilon_0 \partial_t \vec{E}} \\
&= -\frac{1}{2}\frac{1}{\mu_0 c} \partial_t B^2 - \frac{1}{c} \vec{E} \cdot \vec{J} - \frac{\epsilon_0}{2c} \partial_t E^2 \\
&= -\frac{1}{c} \vec{E} \cdot \vec{J} - \frac{\epsilon_0}{2c} \frac{\partial}{\partial t} \left[E^2 + c^2 B^2 \right] \\
&= -\frac{1}{c} \vec{E} \cdot \vec{J} - \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} \, ,
\end{aligned}
\tag{38}
$$

and conclude that

$$
f^0 = -\partial_\nu T^{0\nu} = -\partial_0 T^{00} - \partial_i T^{0i} = \frac{1}{c} \vec{E} \cdot \vec{J} \, , \tag{39}
$$

which is the Joule power per unit volume.

Now, for the spatial components we first compute

$$
\begin{aligned}
\partial_j \tau^{ij} &= \partial_j \left[\epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)\right] \\
&= \epsilon_0 \left[\left(\partial_j E_i\right) E_j + E_i \nabla \cdot \vec{E} - E_j \partial_i E_j\right] \\
&+ \mu_0 \left[\left(\partial_j B_i\right) B_j + B_i \nabla \cdot \vec{B} - B_j \partial_i B_j\right] \\
&= \epsilon_0 \left[E_i \nabla \cdot \vec{E} - E_j \underbrace{\left(\partial_i E_j - \partial_j E_i\right)}_{\epsilon_{ijk}\left(\nabla \wedge \vec{E}\right)_k}\right] \\
&+ \mu_0 \left[B_i \underbrace{\nabla \cdot \vec{B}}_{0} - B_j \underbrace{\left(\partial_i B_j - \partial_j B_i\right)}_{\epsilon_{ijk}\left(\nabla \wedge \vec{B}\right)_k}\right] \\
&= \epsilon_0 \left[E_i \frac{\rho}{\epsilon_0} + \epsilon_{ijk} E_j \frac{\partial B_k}{\partial t}\right] - \frac{1}{\mu_0}\epsilon_{ijk} B_j \mu_0 \left[J_k + \epsilon_0 \frac{\partial E_k}{\partial t}\right] \\
&= \rho E_i - \left(\vec{J} \wedge \vec{B}\right)_i + \epsilon_{ijk}\left(E_j \frac{\partial B_k}{\partial t} - B_j \frac{\partial E_k}{\partial t}\right) \\
&= \rho E_i + \left(\vec{J} \wedge \vec{B}\right)_i + \epsilon_0 \frac{\partial}{\partial t}\left(\vec{E} \wedge \vec{B}\right)_i \\
&= \rho E_i + \left(\vec{J} \wedge \vec{B}\right)_i + \frac{1}{c^2}\frac{\partial}{\partial t} S_i .
\end{aligned}
\tag{40}
$$

Then we can set $\mu = i$ in the stress tensor divergence and get

$$
\begin{aligned}
f^i = -\partial_\nu T^{i\nu} &= -\partial_0 T^{i0} + \partial_j \tau^{ij} \\
&= -\frac{1}{c^2}\frac{\partial}{\partial t} S_i + \rho E_i + \left(\vec{J} \wedge \vec{B}\right)_i + \frac{1}{c^2}\frac{\partial}{\partial t} S_i \\
&= \rho E_i + \left(\vec{J} \wedge \vec{B}\right)_i
\end{aligned}
\tag{41}
$$

which is the Lorentz force per unit volume. Therefore the components vector f^μ are the power transmitted to the sources and the force exerted on them by the electromagnetic fields.

Method 2:

We have :

$$
\begin{aligned}
\partial_\mu T^{\mu\nu} &= \varepsilon_0 \left(\partial_\mu \left(F^\mu_{\ \alpha} F^{\nu\alpha}\right) - \frac{1}{4}\eta^{\mu\nu}\partial_\mu \left(F_{\alpha\beta} F^{\alpha\beta}\right)\right) \\
&= \varepsilon_0 \left((\partial_\mu F^\mu_{\ \alpha}) F^{\nu\alpha} + F^\mu_{\ \alpha} \partial_\mu F^{\nu\alpha} - \frac{1}{2}\partial^\nu F_{\alpha\beta} F^{\alpha\beta}\right) \\
&= \varepsilon_0 \left(-\frac{1}{c\varepsilon_0} J_\alpha F^{\nu\alpha} + F^\mu_{\ \alpha} \partial_\mu F^{\nu\alpha} - \frac{1}{2}\partial^\nu F_{\alpha\beta} F^{\alpha\beta}\right).
\end{aligned}
\tag{42}
$$

Where in the last line, we used one of Maxwell equation. We can show that the last 2 terms are zero. To do so, we need to express terms of the form $\partial_\mu F^{\nu\alpha}$. We can now use the second Maxwell equation $\partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$. This equation implies the Bianchi identity :

$$
\partial^\nu F^{\alpha\beta} + \partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha} = 0 \tag{43}
$$

which allows us to express $\partial^\nu F^{\alpha\beta} = -\partial^\alpha F^{\beta\nu} - \partial^\beta F^{\nu\alpha}$. Using this:

$$
\begin{aligned}
F^\mu{}_\alpha \partial_\mu F^{\nu\alpha} - \frac{1}{2}\partial^\nu F_{\alpha\beta}F^{\alpha\beta} &= F^\mu{}_\alpha \partial_\mu F^{\nu\alpha} + \frac{1}{2}\partial^\alpha F^{\beta\nu}F_{\alpha\beta} + \frac{1}{2}\partial^\beta F^{\nu\alpha}F_{\alpha\beta} \\
&= (\partial^\mu F^{\nu\alpha})F_{\mu\alpha} + \frac{1}{2}(\partial^\alpha F^{\beta\nu})F_{\alpha\beta} + \frac{1}{2}(\partial^\beta F^{\alpha\nu})F_{\beta\alpha} \\
&= 0.
\end{aligned} \tag{44}
$$

Remember that contracted indices are dummy variables : one can rename them without changing the value of the expression. You can see that the second line is the sum of three identical terms with different dummy indices. So we have

$$
\partial_\mu T^{\mu\nu} = -\frac{1}{c}J_\alpha F^{\nu\alpha} = \frac{1}{c}J_\alpha F^{\alpha\nu}\,. \tag{45}
$$

and we can conclude that

$$
f^\mu = \frac{1}{c}J_\alpha F^{\mu\alpha}\,. \tag{46}
$$

Let us now see what the spacial and time component of this equation represent:

$$
f^0 = \frac{1}{c}J_\alpha F^{0\alpha} = \frac{1}{c}J_0 F^{00} + \frac{1}{c}J_i F^{0i} = \frac{1}{c}\mathbf{J}\cdot\mathbf{E} \tag{47}
$$

$$
f^i = \frac{1}{c}J_\alpha F^{i\alpha} = \frac{1}{c}J_0 F^{i0} + \frac{1}{c}J_i F^{ij} = \rho E_i + (\vec{J}\wedge\vec{B})_i\,. \tag{48}
$$

and are respectively the Joule power and Lorentz force per unit volume. Therefore the components vector f^μ are the power transmitted to the sources and the force exerted on them by the electromagnetic fields.

f^μ is a 4−vector:
Clearly f^μ is also a 4−vector since it is the result of a contraction between a 2−tensor and the 4−derivative (or equivalently using (46)). Explicitly, we can show that it transforms as a 4−vector: under a Lorentz transformation Λ, we have

$$
\begin{aligned}
f'^\mu &= -\partial'_\nu T'^{\mu\nu} \\
&= -\Lambda_\nu{}^\alpha \partial_\alpha \Lambda^\mu{}_\beta \Lambda^\nu{}_\gamma T^{\beta\gamma} \\
&= -\Lambda^\mu{}_\beta \Lambda^\nu{}_\gamma \Lambda_\nu{}^\alpha \partial_\alpha T^{\beta\gamma} \\
&= -\Lambda^\mu{}_\beta \underbrace{(\Lambda^{-1})_\gamma{}^\nu \Lambda_\nu{}^\alpha}_{\delta^\alpha_\gamma} \partial_\alpha T^{\beta\gamma} \\
&= -\Lambda^\mu{}_\beta \partial_\alpha T^{\beta\alpha} \equiv \Lambda^\mu{}_\beta f^\beta.
\end{aligned} \tag{49}
$$

(c) The electromagnetic wave with amplitude $E = cB$ can be written as $\vec{E} = \vec{\mathcal{E}}e^{ik\cdot x} + cc.$ with $|\vec{\mathcal{E}}| = E$. The power absorbed by the light sail coincides with the flux of the Poynting vector $\vec{S}$ through its surface, and since $\vec{S}$ is constant all along it, the power per unit area per unit time really just coincides with its modulus:

$$
w_{inc} = \langle\vec{S}\rangle = c\epsilon_0\langle\vec{E}^2\rangle = 2c\epsilon_0 E^2. \tag{50}
$$

For the reflected wave, the computation is exactly the same, since the wave is perfectly reflected, and so the total power is

$$w_{tot} = w_{inc} + w_{ref} = 4c\epsilon_0 E^2. \tag{51}$$

In particular, it is important to notice that the total energy transmitted to the light sail it is *not* the flux of the sum of the Poynting vectors (which would be zero). Indeed, electromagnetic fields sum linearly, but $\vec{S}$ does not, since it is a quadratic for of E and B, so the two contributions need to be separately accounted for.

(d) **Method 1:**

We choose a cartesian system so that the spaceship and the electromagnetic waves are travelling along the $\hat{x}$ axis. This way it is enough to compute the ratio between S'_x and S_x, or equivalently $\frac{T'^{01}}{T^{01}}$, as we have seen in point (a). Note that as we are considering plane-wave, we do not have to consider the ratio of the time averaged quantity $\langle \dots \rangle$ as the overall prefactor cancels out.

The two reference frames $\mathcal{R}$ and $\mathcal{R}'$ are related by a boost along the $\hat{x}$ axis of speed $-\beta$, whose Lorentz matrix is

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{52}$$

The T'^{01} component of stress tensor in $\mathcal{R}'$ is related to those of $\mathcal{R}$ as:

$$\begin{aligned} T'^{01} &= \Lambda^0{}_{\mu}\Lambda^1{}_{\nu}T^{\mu\nu} \\ &= \gamma\left(\Lambda^1{}_{\nu}T^{0\nu} + \beta\Lambda^1{}_{\nu}T^{1\nu}\right) \\ &= \gamma^2\left(\beta T^{00} + T^{01} + \beta^2 T^{10} + \beta T^{11}\right). \end{aligned} \tag{53}$$

For a plane wave the components of the stress tensor simplify quite a bit, since $F^{\alpha\beta}F_{\alpha\beta} = 2(c^2B^2 - E^2) = 0$ and $E_1 = B_1 = 0$ (the fields are orthogonal to the direction of propagation), so

$$T^{00} \overset{(35)}{=} \frac{\epsilon_0}{2}\left(c^2\vec{B}^2 + \vec{E}^2\right) = \epsilon_0 E^2, \tag{54}$$

$$T^{01} \overset{(36)}{=} \frac{S^i}{c} = \epsilon_0 E^2, \tag{55}$$

$$T^{11} \overset{(37)}{=} \epsilon_0\left(-E_1E_1 + \frac{1}{2}\delta_{11}\vec{E}^2\right) + \frac{1}{\mu_0}\left(-B_1B_1 + \frac{1}{2}\delta_{11}\vec{B}^2\right) = \epsilon_0 E^2. \tag{56}$$

Finally,

$$T'^{01} = \gamma^2(1+\beta)^2\epsilon_0 E^2 \implies \frac{T'^{01}}{T^{01}} = \gamma^2(1+\beta)^2 = \frac{3}{2}. \tag{57}$$

Method 2:

From the answer of (c), we know that $w \propto E^2$ and so it is enough to compute $|\vec{E}'| = |\vec{E}_\perp|$ (we also have $\vec{E} = \vec{E}_\perp$). The boosted electric field is

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \boldsymbol{\beta} \wedge \vec{B}) \tag{58}$$

For a plane wave, we have $\vec{B} = \frac{1}{c}\vec{n} \wedge \vec{E}$ where $\vec{n}$ is a unit vector along the direction of the propagation of the wave. Thus

$$\begin{aligned} \vec{E}'_\perp &= \gamma(\vec{E}_\perp + \boldsymbol{\beta} \wedge (\vec{n} \wedge \vec{E}_\perp)) && (59)\\ &= \gamma(\vec{E}_\perp + (\boldsymbol{\beta} \cdot \vec{E}_\perp)\vec{n} - (\boldsymbol{\beta} \cdot \vec{n})\vec{E}_\perp) && (60)\\ &= \gamma(1+\beta)\vec{E}_\perp && (61) \end{aligned}$$

where we used that $\boldsymbol{\beta} = -\beta\vec{n}$. It is now straightforward to compute the power ratio

$$\frac{w'}{w} = \frac{|\vec{E}'|^2}{|\vec{E}|^2} = \frac{|\vec{E}'_\perp|^2}{|\vec{E}_\perp|^2} = \gamma^2(1+\beta)^2 = \frac{3}{2}\,. \tag{62}$$