Classical Electrodynamics

Week 12

1. *Symmetries of the wave equation – Lorentz Boost*
During the lecture, you derived the symmetries of the wave equation using *Wick* rotation. Here, you will rederive them independently. For simplicity, we will consider only one spacial dimension to focus on the symmetry involving space and time.

The d'Alembertian operator can be written in matrix notation

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta} \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix}, \tag{1}$$

and the most general linear transformation of spacetime is

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} \tag{2}$$

a) Suppose that $\Psi(x,t)$ is a solution of $\Box\Psi(x,t) = 0$. What if the constraints on Λ such that $\Psi'(x,t) = \Psi(x',t')$ is also a solution?

b) Show that the constraints imply that

$$\Lambda = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \tag{3}$$

with χ a free parameter.

c) By studying the trajectory of the origin $x' = 0$, relate the rapidity χ to $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, where v is the velocity of $\mathcal{O}'$.

d) Write Λ in term of β, γ. Do you recognize the familiar Lorentz boost?

Solutions

a) We are asked to show that $\Box\Psi'(x,t) = 0$ knowing that $\Box\Psi(x,t) = 0$ (The latter implies that $\Box'\Psi(x',t') = 0$).

We can straightforwardly relate the derivatives

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \Lambda_{22}\frac{\partial}{\partial x'} + \Lambda_{12}\frac{1}{c}\frac{\partial}{\partial t'} \tag{4}$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{1}{c}\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = \Lambda_{21}\frac{\partial}{\partial x'} + \Lambda_{11}\frac{1}{c}\frac{\partial}{\partial t'} \tag{5}$$

and we obtain

$$\begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix} = \Lambda^T \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t'} \\ \frac{\partial}{\partial x'} \end{pmatrix}. \tag{6}$$

Thus we have

$$\Box\Psi'(x,t) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)\eta\begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t}\\ \frac{\partial}{\partial x}\end{pmatrix}\Psi'(x,t) \tag{7}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}\right)\Lambda\eta\Lambda^T\begin{pmatrix}\frac{1}{c}\frac{\partial}{\partial t'}\\ \frac{\partial}{\partial x'}\end{pmatrix}\Psi(x',t') \tag{8}$$

If $\eta = \Lambda\eta\Lambda^T$ then the right hand side become $\Box'\Psi(x',t') = 0$. The symmetries are thus generates by all transformations such that $\eta = \Lambda\eta\Lambda^T$.

b) In component, $\eta = \Lambda\eta\Lambda^T$ is

$$\begin{cases} -1 = -\Lambda_{11}^2 + \Lambda_{12}^2 \\ \;\;\,1 = -\Lambda_{21}^2 + \Lambda_{22}^2 \\ \;\;\,0 = \Lambda_{11}\Lambda_{21} - \Lambda_{12}\Lambda_{22} \end{cases} \tag{9}$$

which is solved by

$$\Lambda = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \tag{10}$$

with χ a free parameter.

c) The trajectory described by the $x' = 0$ in the (x,t) coordinate correspond to the trajectory of the reference frame $\mathcal{O}'$ seen by $\mathcal{O}$.

$$0 = -ct\sinh\chi + x\cosh\chi \tag{11}$$

$$v \equiv \frac{x}{t} = c\tanh\chi \tag{12}$$

So we get

$$\beta = \frac{v}{c} = \tanh\chi\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\chi \tag{13}$$

d) We then obtain

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \tag{14}$$

And in component we see that

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ \;x' = \gamma(x - \beta ct) \end{cases} \tag{15}$$

the familiar Lorentz transformation.

2. *Invariance of Electric potential*
Consider a scalar potential ϕ.

a) In electrostatics in the absence of charges, the scalar potential ϕ satisfies the Laplace equation $\Delta\phi = 0$. Find the set of transformations of space that leave this equation invariant.

b) In electrodynamics in Lorenz gauge, the scalar potential satisfies the wave equation $\Box\phi = 0$. Find the set of transformations of space and time that leave this equation invariant.
Note: This exercise is similar in spirit as 1a). Here try to derive to work in tensor notation and in $3+1$ dimensions.

Solution

a) The translation

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \mathbf{a}\,, \tag{16}$$

where $\mathbf{a}$ is a constant, is a transformation that leaves the Laplace equation invariant. This feature is pretty easy to show: we can see that the derivative does not change under a translation:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial}{\partial x'}\,. \tag{17}$$

Therefore, we have

$$\Delta = \Delta'\,. \tag{18}$$

Now, we can try with the rotation. As we have seen during the course of linear algebra, the rotation can be defined with an orthogonal matrix O:

$$\mathbf{x} \to \mathbf{x}' = O\mathbf{x}\,, \tag{19}$$

with $O^T O = \mathbb{I}$. The rotation is determined by three parameters. Indeed, two parameters are needed to select the axis around which the system will rotate and one parameter determines the angle of rotation. Since the relation between $\mathbf{x}$ and $\mathbf{x}'$ is linear (O does not depend on x), we can see that

$$\frac{\partial x'_i}{\partial x_j} = O_{ij}\,. \tag{20}$$

Using the chain rule, we can find the relation between the derivative with respect to the old and the new variables:

$$\frac{\partial}{\partial x_j} = \sum_i \frac{\partial x'_i}{\partial x_j}\frac{\partial}{\partial x'_i} \equiv \sum_i O_{ij}\frac{\partial}{\partial x'_i}\,. \tag{21}$$

In vector notation, (21) can be written in the following way:

$$\nabla_x \equiv O^T \nabla_{x'}\,, \tag{22}$$

and the direct implication is

$$\nabla_{x'} \equiv O\nabla_x\,, \tag{23}$$

where we have just applied O on both sides of (22). The Laplacian Δ can be written as

$$\Delta = \nabla_x^T \nabla_x = \nabla_x^T \mathbb{I} \nabla_x\,. \tag{24}$$

Using (23), we can find the new Laplacian:

$$\Delta' = \nabla_{x'}^T \nabla_{x'} = (O\nabla_x)^T O\nabla_x = \nabla_x^T O^T O \nabla_x = \nabla_x^T \mathbb{I} \nabla_x = \Delta\,. \tag{25}$$

Therefore, also the rotation leaves the the Laplace equation invariant. The translation and the rotation together form a group of six parameters that is called Euclidean group in three dimensions. [1]

b) Let us define for simplicity the four dimensional vector $\tilde{\mathbf{x}} = (ct, \mathbf{x})$. A generic affine transformation of spacetime can be defined by a matrix $\Lambda \in \mathbb{R}^{4\times 4}$ and a vector $\tilde{\mathbf{a}} \in \mathbb{R}^4$ through:

$$\tilde{\mathbf{x}} \to \tilde{\mathbf{x}}' = \Lambda\tilde{\mathbf{x}} + \tilde{\mathbf{a}} \tag{26}$$

The D'Alambert operator is trivially invariant under the class of inhomogeneous transformations:

$$\tilde{\mathbf{x}} \to \tilde{\mathbf{x}}' = \tilde{\mathbf{x}} + \tilde{\mathbf{a}} \tag{27}$$

corresponding to translations in space and time.

We then need to find the class of matrices Λ such that the D'Alambert operator is invariant under the homogeneous transformation:

$$\tilde{\mathbf{x}} \to \tilde{\mathbf{x}}' = \Lambda\tilde{\mathbf{x}} \tag{28}$$

We can define the vector of derivatives $\tilde{\partial} = \begin{pmatrix} \frac{1}{c}\partial_t \\ \nabla_{\mathbf{x}} \end{pmatrix}$ and write the D'Alambert operator as

$$\Box = \tilde{\partial}^T \,\eta\, \tilde{\partial}$$

where η is the Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

By applying the chain rule we notice that:

$$(\tilde{\partial})_\mu = \frac{\partial}{\partial \tilde{x}^\mu} = \frac{\partial \tilde{x}'^\nu}{\partial \tilde{x}^\mu}\frac{\partial}{\partial \tilde{x}'^\nu} = \Lambda^\nu{}_\mu (\tilde{\partial}')_\nu \qquad \mu = 0, 1, 2, 3 \qquad \Rightarrow \qquad \tilde{\partial} = \Lambda^T \tilde{\partial}'$$

By imposing invariance we then obtain:

$$\Box = \tilde{\partial}^T \,\eta\, \tilde{\partial} = \tilde{\partial}'^T \Lambda\,\eta\,\Lambda^T \tilde{\partial}' = \Box' = \tilde{\partial}'^T \,\eta\, \tilde{\partial}'$$

This results into the condition:

$$\Lambda\,\eta\,\Lambda^T = \eta \tag{29}$$

The matrices satisfying (29) form a group noted $O(1, 3)$, the Lorentz group. We also define the proper orthochronous Lorentz group, noted $SO^+(1, 3)$, as the subgroup of the Lorentz group such that $\det(\Lambda) = +1$ and $\Lambda_{00} \geq 1$: this subgroup is connected and every element of the Lorentz group can be obtained by combining an element of $SO^+(1, 3)$ and an element of the discrete group $\{\mathbb{I}, P, T, PT\}$, where $P = \eta$, $T = -\eta$.

The Lorentz group has six parameter (eq. (29) gives 10 constraints, since $\Lambda\,\eta\,\Lambda^T$ is a symmetric matrix): three of them define the spacial rotations, three the boosts, one for each independent direction.

The Lorentz group together with spacetime translations forms the Poincaré group (ten parameters).

[1]Dilatations $\mathbf{x} \to \lambda\mathbf{x}$ is also a symmetry of the Laplace equation. However, we do not consider it because it is usually broken by the presence of sources.

3. Determine the potentials $(\phi, \mathbf{A})$ of a charge q at rest at the origin of a reference frame $\mathcal{R}$. Then consider a reference frame $\mathcal{R}'$ moving with uniform velocity $\mathbf{v}$ with respect to $\mathcal{R}$. What are the electromagnetic potentials $(\phi', \mathbf{A}')$ in the reference frame $\mathcal{R}'$? Compare your results with the Liénard-Wiechert potentials.

Solution

In the reference frame $\mathcal{R}$, a charge q is sitting at rest at the origin. The potentials are the usual expressions:

$$\begin{cases} \phi = \frac{q}{4\pi\epsilon_0 r} \\ \mathbf{A} = 0 \end{cases} . \tag{30}$$

Let us move to the reference frame $\mathcal{R}'$ moving with uniform velocity $\mathbf{v}$ chosen to be in the x direction with respect to $\mathcal{R}$. We know that the potentials $A^\mu = (\phi, c\mathbf{A})$ transform as a Lorentz four-vector:

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x') = \Lambda^\mu{}_\nu A^\nu(x) \tag{31}$$

where Λ is the matrix of the boost in the x direction:

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{32}$$

Using this, and the coordinate relation $x = \gamma(x' + \beta ct'), y = y', z = z'$, one can write:

$$\phi'(t', \mathbf{x}') = \gamma\phi(t, \mathbf{x}) - \beta c\gamma \mathbf{A}_x(t, \mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{\gamma}{\sqrt{\gamma^2(x' + \beta ct')^2 + y'^2 + z'^2}} \tag{33}$$

and for the vector potential:

$$\mathbf{A}'(t', \mathbf{x}') = -\frac{q}{4\pi\epsilon_0 c} \frac{\beta\gamma}{\sqrt{\gamma^2(x' + \beta ct')^2 + y'^2 + z'^2}} \mathbf{e}_x . \tag{34}$$

In the frame $\mathcal{R}'$, the charge is moving at a constant velocity $\mathbf{v} = -v\mathbf{e}_x$. We find exactly the same result as the Liénard-Wiechert formula that we derived in exercise 1 of Week 5, but with considerably less effort !

4. The predictions of special relativity are often counter-intuitive because our daily experience is limited to velocities much smaller than the speed of light. The goal of this exercise is to develop your relativistic intuition through the discussion of several thought experiments.

Suggestion: You are not expected to do long computation. Reflect on the situation and draw a spacetime diagram describing each experiment.

a) Two identical rockets are at rest connected by a rope of length 100 m. The rope is stretched and cannot be extended beyond 101 m without breaking. The two rockets are programmed to start accelerating exactly at the same time along the direction parallel to the rope. They stop accelerating when they reach the velocity 0.8c. Did the rope break?

b) John wants to put a 10 m long ladder inside his garage, which also has length 10 m. The garage has doors at both ends that can be closed simultaneously by pressing a button in the middle of the garage. Paul tells John to run with the ladder towards the garage while he stays next to the button. Since John can run at the speed $0.8c$ the ladder will be Lorentz contracted and Paul will easily be able to close the garage doors with the ladder inside. On the other hand, for John it is the garage that gets Lorentz contracted and therefore it will not be possible to close the garage doors. Who is right?

c) Consider two LED's and two photo-detectors placed at the four vertices of a vertical square of side 10 m. The LEDs are placed in the bottom vertices of the square and the photo-detectors on the top vertices. Each LED emits light towards the photo-detector just above it (on the the same side of the square) and the two photo-detectors are connected to an electronic NOR that will ring an alarm if and only if none of the photo-detectors detects light at the same time. Now consider an opaque bar of length 10 m that crosses the square horizontally at the speed $c/2$ (and at mid height). From the point of view of the square the bar gets Lorentz contracted and will not be able to block the light of both LEDs at the same time. However, from the point of view of the bar, the square gets Lorentz contracted and both LEDs will be blocked at the same time during some time. Will the alarm ring?

Solution

a) The key point of this problem is that the two rockets remain at distance $L = 100$ m in the laboratory frame during the whole motion. Indeed, they are two independent objects with identical laws of motion. If we call $x_1(t)$ and $x_2(t)$ the law of motion of the first and the second rocket respectively, we may write

$$x_1(t) = x_1(0) + f(t) \tag{35}$$
$$x_2(t) = x_1(0) + L + f(t), \tag{36}$$

where $f(t)$ is a function describing the motion of the object, with the properties that $f(0) = 0$ and $f(t) = 0.8c\,(t - t^*) + f(t^*)$ for $t > t^*$. The distance between the rockets in the laboratory frame is $x_2(t) - x_1(t) = L$ at all times, as advertised. Now, let us consider the situation at some time $t > t^*$. The rockets are moving at constant velocity, and we can go to the rest frame via a Lorentz transformation. Because of length contraction, the distance between the rockets is γL in this frame. The resistance of the rope to stretching is defined at rest, and at the end of the period of acceleration the rope should be γ times longer than at the beginning in order to be still attached to both rockets. In conclusion, if the rope cannot be extended, it did break at some point during the motion.

b) As we know, if the ladder is of length l_1 in John's reference frame, it will be of length $l_2 = \frac{l_1}{\gamma}$ in the reference frame of the garage, therefore for an observer at rest with respect to Paul the ladder is shorter than the garage. The opposite is true for an observer that is instead at rest with respect to John. In that case, the Lorentz contraction must be applied to the garage and not to the ladder.

The key point to understand this apparent contradiction is that in special relativity two events that happen at the same time in a given reference frame do not happen at the same time in another reference frame. In our particular case, the doors of the garage close at the same time only in the reference frame of Paul. In the reference frame of John, the time interval between the closing of each door is

$$\Delta t' \equiv t'_2 - t'_1 = -\gamma \frac{v}{c^2}(x_2 - x_1) = -\gamma \frac{v}{c^2} \Delta x \ , \tag{37}$$

where Δx is the length of the garage at rest. Therefore, the exit door closes earlier than the entrance door. In the time $|\Delta t'|$ the extremes of the ladder cover the distance

$$d_1 = v|\Delta t'| = \gamma \frac{v^2}{c^2} \Delta x \,. \tag{38}$$

This distance is larger than the distance between the back of the ladder and the entrance door at the moment when the front of the ladder reaches the exit door, that is

$$d_1 > d_2 = l_1 - \frac{\Delta x}{\gamma} \,. \tag{39}$$

So if the exit door closes when the front of the ladder arrives, the back of the ladder has time to reach the entrance door before it closes. One can check that condition (39) is precisely equivalent to $\Delta x > l_1/\gamma$.

c) The alarm will ring if the two photo-detectors don't receive any signal *simultaneously* in the frame where they are at rest (where the measurement is taken).

Let's call this frame $\mathcal{R}$ and its spacetime coordinates (ct, x). As anticipated in the text of the exercise, in this frame the bar gets Lorentz contracted and we have:

$$L_{frame} = L_0 = 10\,\mathrm{m} \qquad\qquad L_{bar} = \frac{L_0}{\gamma(v)} < L_0$$

so that there exists no time t such that both LEDs will be blocked. Considering what we stated above, the alarm will not go off.

Let us now consider the frame $\mathcal{R}'$ where the bar is at rest and its spacetime coordinates (ct', x'). In this frame we have:

$$L_{bar} = L_0 = 10\,\mathrm{m} \qquad\qquad L_{frame} = \frac{L_0}{\gamma(v)} < L_0$$

There is then a time interval $\delta t' = \frac{L_0(1-1/\gamma)}{c/2}$ such that both LED's will be blocked. What seems like a contradiction is easily explained by looking at the spacetime diagram in 1: in the frame where the bar is at rest (green area), the square frame crosses the bar from left to right (red area) at $v = -0.5\,c$. The time interval T corresponds to $t_A = t_B < t' < t_D = t_E$. Lines joining events that are simultaneous in $\mathcal{R}'$ are parallel to the x' axis, lines joining events that are simultaneous in $\mathcal{R}$ are parallel to the x axis: we can see that while event A is simultaneous to B in $\mathcal{R}'$, it is simultaneous to C in $\mathcal{R}$, and since $t'_F < t'_B$, the two extremities of the bar never simultaneously cover the two photo-detectors in $\mathcal{R}$ as we had concluded in the beginning.

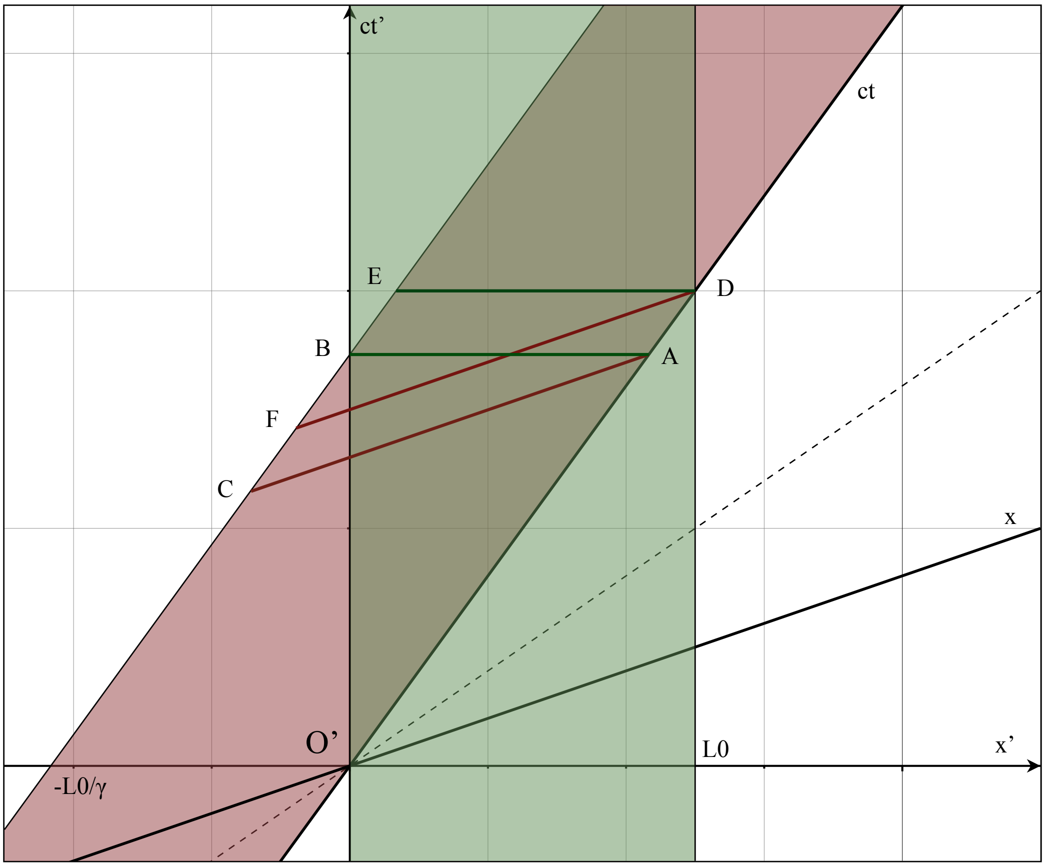

Figure 1: Spacetime diagram of the LED thought experiment.