Classical Electrodynamics

Week 10

1. *Velocity's transformations in Special Relativity*
In this exercise, you will study how velocities v_1 and v_2 transform from one reference frame $\mathcal{R}_1$ to another $\mathcal{R}_2$ in the context of Special Relativity, in different scenarios.

a) The reference frame $\mathcal{R}_1$ is moving along the x-axis with speed v_0 in the reference frame of the laboratory $\mathcal{R}_0$, and a particle is moving with speed v_1 along the x-axis in the reference frame $\mathcal{R}_1$. What is the velocity of the particle in the reference frame $\mathcal{R}_0$?

b) Two particles (with the same mass) are moving in the same direction with velocities v_1 and $v_2 > v_1$ in the reference frame $\mathcal{R}_1$. At what speed should the reference frame $\mathcal{R}_2$ move with respect to $\mathcal{R}_1$ so that the center of mass condition $v'_1 + v'_2 = 0$ is obeyed? What is the value of v'_1? Is your result compatible with your non-relativistic intuition?

c) The trajectory of a particle moving at constant velocity makes an angle θ with the x-axis of a reference frame $\mathcal{R}_1$. Compute the corresponding angle θ' in a reference frame $\mathcal{R}_2$ moving with speed v along the x-axis of $\mathcal{R}_1$.

d) Consider two particles (with the same mass) moving at the same speed v. The angle between their trajectories is θ. Find a reference frame in which $\mathbf{v}'_1 + \mathbf{v}'_2 = 0$.

Solution

a) **First method** We will write $\tilde{\mathbf{x}} = (ct, x)$ the coordinates in $\mathcal{R}_0$ and $\tilde{\mathbf{x}}' = (ct', x')$ the coordinates in $\mathcal{R}_1$. The particle is moving with speed v_1 along x in $\mathcal{R}_1$ so by definition:

$$v_1 = \frac{dx'}{dt'} \tag{1}$$

and we want to express the speed of the particle in $\mathcal{R}_0$ which is:

$$v = \frac{dx}{dt}. \tag{2}$$

We want to express (ct, x) as function of (ct', x') so let's write the change of coordinates from $\mathcal{R}_1$ to $\mathcal{R}_0$. The laboratory frame is moving along the x-axis with velocity $-v_0$ in the frame $\mathcal{R}_1$ so:

$$\begin{cases} ct = \gamma_0(ct' + \beta_0 x') \\ x = \gamma_0(x' + \beta_0 ct') \end{cases} \tag{3}$$

where

$$\beta_0 = \frac{v_0}{c} \quad \text{and} \quad \gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}}. \tag{4}$$

Then we have

$$\begin{aligned}\frac{dx}{dt} =& c\frac{dx' + \beta_0 c dt'}{cdt' + \beta_0 dx'} = c\frac{v_1 + c\beta_0}{c + \beta_0 v_1} \\ =& \frac{v_1 + v_0}{1 + \frac{v_1 v_0}{c^2}}.\end{aligned} \tag{5}$$

Notice that in the non-relativistic limit, this reduces to the usual formula $v = v_0 + v_1$.

Second method Let us call $\mathcal{R}_2$ the rest frame of the particle. $\mathcal{R}_2$ is moving along the x-axis with speed v_1 in the frame $\mathcal{R}_1$ and we want to express its speed v in the frame $\mathcal{R}_0$.

We can express $\tilde{\mathbf{x}}'' = (ct'', x'')$, the coordinates in $\mathcal{R}_2$, as function of coordinates of $\mathcal{R}_0$ by making two successive boosts, the first of velocity v_0 to go from $\mathcal{R}_0$ to $\mathcal{R}_1$, the second of velocity v_1 to go from $\mathcal{R}_1$ to $\mathcal{R}_2$. In matrix notation (keeping only the (ct, x) coordinates), the transformations are defined as

$$\tilde{\mathbf{x}}' = \Lambda(v_0)\tilde{\mathbf{x}}, \quad \tilde{\mathbf{x}}'' = \Lambda(v_1)\tilde{\mathbf{x}}' \quad \Rightarrow \quad \tilde{\mathbf{x}}'' = \Lambda(v_1)\Lambda(v_0)\tilde{\mathbf{x}}\,. \tag{6}$$

The transformation can be written as:

$$\begin{aligned}\Lambda(v_1)\Lambda(v_0) =& \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_0 & -\beta_0\gamma_0 \\ -\beta_0\gamma_0 & \gamma_0 \end{pmatrix} \\ =& \begin{pmatrix} \gamma_0\gamma_1(1+\beta_0\beta_1) & -\gamma_0\gamma_1(\beta_0+\beta_1) \\ -\gamma_0\gamma_1(\beta_0+\beta_1) & \gamma_0\gamma_1(1+\beta_0\beta_1) \end{pmatrix}.\end{aligned} \tag{7}$$

But Lorentz transformation form a group, which means that:

$$\Lambda(v_1)\Lambda(v_0) = \Lambda(v) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}, \tag{8}$$

which corresponds to doing one boost from the frame $\mathcal{R}_0$ to $\mathcal{R}_2$

$$\tilde{\mathbf{x}}'' = \Lambda(v)\tilde{\mathbf{x}}. \tag{9}$$

We need to solve only one component of the matrix equation. For example, writing $\Lambda(v)_{11}$ using equations (8) and (7) one has:

$$\gamma = \gamma_0\gamma_1(1+\beta_0\beta_1). \tag{10}$$

After inverting and squaring this reduces to:

$$1 - \frac{v^2}{c^2} = \frac{\left(1 - \frac{v_0^2}{c^2}\right)\left(1 - \frac{v_1^2}{c^2}\right)}{\left(1 + \frac{v_0 v_1}{c^2}\right)^2}, \tag{11}$$

and finally

$$v = \frac{v_1 + v_0}{1 + \frac{v_1 v_0}{c^2}}. \tag{12}$$

b) The reference frame $\mathcal{R}_2$ is boosted with respect to $\mathcal{R}_\infty$ by a velocity v in the same direction as v_1 and v_2. The velocity of the particles in the reference frame $\mathcal{R}_2$ was derived in the previous point:

$$v'_{1,2} = \frac{v_{1,2} - v}{1 - \frac{vv_{1,2}}{c^2}}. \tag{13}$$

We find v through the equation $v'_1 + v'_2 = 0$, which gives:

$$\frac{\beta_1 - \beta}{1 - \beta_1 \beta} + \frac{\beta_2 - \beta}{1 - \beta_2 \beta} = 0\,, \tag{14}$$

where, as usual, we denote $\beta_i = v_i/c$ the velocities measured in units of c. This gives the following second order equation:

$$\beta^2 - 2\beta \left(\frac{1 + \beta_1 \beta_2}{\beta_1 + \beta_2} \right) + 1 = 0, \tag{15}$$

whose solutions are

$$\beta_\pm = \frac{1}{\beta_1 + \beta_2} \left(1 + \beta_1 \beta_2 \pm \sqrt{(1 - \beta_1^2)(1 - \beta_2^2)} \right). \tag{16}$$

It is not difficult to show that the plus sign leads to an unphysical velocity $\beta > 1$, therefore we choose the minus sign. Equivalently, one can notice that, when $\beta_1 \to -\beta_2$, $\beta_+ \to \infty$ while $\beta_- \to 0$, which is the obvious correct result. Plugging the solution back in eq. (13), we get

$$\beta'_1 = \frac{1}{\beta_2 - \beta_1} \left(-1 + \beta_1 \beta_2 + \sqrt{(1 - \beta_1^2)(1 - \beta_2^2)} \right). \tag{17}$$

In the non-relativistic limit, these formulae are reduced to the usual ones:

$$v = \frac{v_1 + v_2}{2}, \qquad v'_1 = \frac{v_1 - v_2}{2}. \tag{18}$$

c) As we know, in the reference frame $\mathcal{R}_2$ the positions and the time are given by

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{19}$$

$$\tag{20}$$

Therefore we have

$$u'_x \equiv \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}\ , \tag{21}$$

and

$$u'_y \equiv \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{vdx}{c^2}\right)} = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}\ . \tag{22}$$

Therefore, the angles θ and θ' are related in the following way:

$$\tan\theta' \equiv \frac{u'_y}{u'_x} = \frac{u_y}{\gamma(u_x - v)} = \frac{u \sin\theta}{\gamma(u\cos\theta - v)}\ . \tag{23}$$

d) Suppose that $\theta < \pi$ and choose a boost of speed v_0 along a direction which lies in the plane of the two trajectories and makes an angle $\theta/2$ with each one of them. We denote by x this direction and by y the orthogonal one in the plane, and we use the result of the previous exercises to write the condition $\mathbf{v}_1' + \mathbf{v}_2' = 0$ in components:

$$\frac{\beta_x - \beta_0}{1 - \beta_x\beta_0} + \frac{\beta_x - \beta_0}{1 - \beta_x\beta_0} = 0 \tag{24}$$

$$\frac{\beta_y}{\gamma\,(1 - \beta_x\beta_0)} - \frac{\beta_y}{\gamma\,(1 - \beta_x\beta_0)} = 0. \tag{25}$$

The second equation is automatically satisfied, as a consequence of our choice for the direction of the boost. The first equation is satisfied by the choice

$$v_0 = v_x = v\cos\frac{\theta}{2}\,. \tag{26}$$

2. *Synchrotron radiation*

Consider a non-relativistic electron ($v \ll c$) in circular movement due to a magnetic field $\mathbf{B}$ orthogonal to the plane of the movement.

a) Calculate the Poynting vector $\mathbf{S} = \epsilon_0 c^2 \mathbf{E}_e \times \mathbf{B}_e$, using the radiative part of the electromagnetic fields $\mathbf{E}_e$, $\mathbf{B}_e$ produced by the accelerated electron. You can use the formulas of Liénard-Wiechert in the non-relativistic limit ($v \ll c$).

b) Calculate the time average of the Poynting vector

$$\langle \mathbf{S} \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \mathbf{S}(t)\,, \tag{27}$$

and determine the total radiated power.

c) Study the angular distribution of the radiation.

Solution

a) Let us use cartesian coordinates where $\mathbf{B}$ points in the z direction: $\mathbf{B} = B_0\hat{z}$. The electron has a uniform circular trajectory of coordinates:

$$\mathbf{r}_0(t) = r_0 \begin{pmatrix} \cos(\omega_s t) \\ \sin(\omega_s t) \\ 0 \end{pmatrix}, \tag{28}$$

where ω_s is the Larmor frequency:

$$\omega_s = \frac{eB_0}{m_e}. \tag{29}$$

Remember the Liénard-Wiechert formula giving the electric and magnetic field produced by a moving point charge:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2}\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\left((\mathbf{n}-\boldsymbol{\beta})(1-\beta^2) + \frac{R}{c}\mathbf{n}\times\left((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right)\right)$$

where all the quantities are evaluated at the retarded time $t' = t - \frac{R}{c}$, R is the distance between the point of observation and the position of the charge $R = |\mathbf{r} - \mathbf{r}_0(t')|$ and $\mathbf{n}$ is the unit vector defined as $\mathbf{n} = \frac{\mathbf{r}-\mathbf{r}_0(t')}{R}$. The magnetic field is given by:

$$\mathbf{B} = \frac{1}{c}\mathbf{n} \times \mathbf{E}. \tag{30}$$

Now we take the non-relativistic limit $\beta \ll 1$ of this formula so we can neglect $\boldsymbol{\beta}$ in front of $\mathbf{n}$ and we consider the radiative regime $R \gg |\mathbf{r}_0|$ so the first term in the parenthesis (the Coulomb term) drops off. We are left with:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{Rc}\mathbf{n} \times \left(\mathbf{n} \times \dot{\boldsymbol{\beta}}\right)$$

Far from the point charge, we have $R = |\mathbf{r} - \mathbf{r}_0(t')| \approx |\mathbf{r}| = r$ and $\mathbf{n} = \frac{\mathbf{r}-\mathbf{r}_0(t')}{R} \approx \frac{\mathbf{r}}{R} \approx \hat{r}$. Doing the cross-products explicitly, we find the formulas:

$$\mathbf{B}_e = \frac{q}{4\pi\epsilon_0 c^3}\frac{\mathbf{a} \times \hat{r}}{r} \qquad \mathbf{E}_e = \frac{q}{4\pi\epsilon_0 c^2}\frac{(\mathbf{a} \times \hat{r}) \times \hat{r}}{r}, \tag{31}$$

where $\mathbf{a}$ is the acceleration of the electron, in this case of a circular motion it is simply $\mathbf{a}(t) = -\omega_s^2\mathbf{r}_0(t)$. In principle, the above formula should be evaluated at the retarded time $t' = t - \frac{r}{c}$ but for fixed r this is just a shift in t so it will not change the following discussion.

The magnetic field is given by:

$$\begin{aligned}\mathbf{B}_e =& \frac{e\omega_s^2 r_0}{4\pi\epsilon_0 c^3 r}\begin{pmatrix}\cos(\omega_s t')\\ \sin(\omega_s t')\\ 0\end{pmatrix} \times \begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}\\ =& \frac{e^3 B_0^2 r_0}{4\pi\epsilon_0 m_e^2 c^3 r}\begin{pmatrix}\sin(\omega_s t')\cos\theta\\ -\cos(\omega_s t')\cos\theta\\ -\sin(\omega_s t' - \varphi)\sin\theta\end{pmatrix}\end{aligned} \tag{32}$$

and the electric field is $\mathbf{E}_e = c\mathbf{B}_e \times \hat{r}$.

The Poynting vector is now:

$$\begin{aligned}\mathbf{S}(t) =& \epsilon_0 c^2 \mathbf{E}_e \times \mathbf{B}_e = \epsilon_0 c^3(\mathbf{B}_e \times \hat{r}) \times \mathbf{B}_e = \epsilon_0 c^3\left(|\mathbf{B}_e|^2\hat{r} - (\mathbf{B}_e \cdot \hat{r})\mathbf{B}_e\right)\\ =& \epsilon_0 c^3|\mathbf{B}_e|^2\hat{r} = \frac{e^6 B_0^4 r_0^2}{16\pi^2\epsilon_0 m_e^4 c^3 r^2}\left(\cos^2\theta + \sin^2(\omega_s t' - \varphi)\sin^2\theta\right)\hat{r}.\end{aligned} \tag{33}$$

We can understand better the term in brackets by noticing that:

$$\left(\cos^2\theta + \sin^2(\omega_s t' - \varphi)\sin^2\theta\right) = |\hat{a}(t') \times \hat{r}|^2 = \sin^2\alpha(t) \tag{34}$$

where $\alpha(t)$ is the angle between $\mathbf{a}(t')$ and $\mathbf{r}$.

b) The time average of the Poynting vector is:

$$\langle\mathbf{S}\rangle = \frac{e^6 B_0^4 r_0^2}{16\pi^2\epsilon_0 m_e^4 c^3 r^2}\hat{r}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \sin^2\alpha(t)dt\right), \tag{35}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \sin^2\alpha(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\cos^2\theta + \sin^2(\omega_s t' - \varphi)\sin^2\theta\right) dt$$
$$= \lim_{T\to\infty}\frac{1}{T}\left[t\cos^2\theta + \sin^2\theta\frac{2(\omega_s t - \varphi) - \sin(2(\omega_s t - \varphi))}{4\omega_s}\right]_{-\frac{r}{c}}^{T-\frac{r}{c}}$$
$$= \cos^2\theta + \frac{1}{2}\sin^2\theta$$

so we get

$$\langle \mathbf{S}\rangle = \frac{e^6 B_0^4 r_0^2}{32\pi^2\epsilon_0 m_e^4 c^3 r^2}(1+\cos^2\theta)\hat{r}. \tag{36}$$

Notice that the Poynting vector (33) depend on θ and φ whereas the time averaged Poynting vector (36) depend only on θ. This is because the radiated field depend on the precise position of the electron but once we average over the periodic trajectory we gain a cylindrical symmetry of axis z.

The Poynting vector is the flux density of radiated energy. The total power is given by the flux of the Poynting vector through a surface enclosing the system that we choose to be a sphere of radius R.

$$\begin{aligned}
P(t) =& \int_S \mathbf{S}(t)\cdot \mathbf{d}\sigma \\
=& \frac{e^6 B_0^4 r_0^2}{16\pi^2\epsilon_0 m_e^4 c^3 R^2}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\, R^2 \left(\cos^2\theta + \sin^2(\omega_s t' - \varphi)\sin^2\theta\right) \\
=& \frac{e^6 B_0^4 r_0^2}{16\pi^2\epsilon_0 m_e^4 c^3}\int_{-1}^{1} d\cos\theta\, 2\pi\cos^2\theta + \sin^2\theta\left[-\frac{2(\omega_s t' - \varphi) - \sin\left(2(\omega_s t' - \varphi)\right)}{4}\right]_0^{2\pi} \\
=& \frac{e^6 B_0^4 r_0^2}{16\pi^2\epsilon_0 m_e^4 c^3}\int_{-1}^{1} d\cos\theta\, \left(2\pi\cos^2\theta + \pi\sin^2\theta\right) \\
=& \frac{e^6 B_0^4 r_0^2}{16\pi\epsilon_0 m_e^4 c^3}\int_{-1}^{1} d\cos\theta\,(1+\cos^2\theta) \\
=& \frac{e^6 B_0^4 r_0^2}{6\pi\epsilon_0 m_e^4 c^3} = \frac{e^2\omega_s^4 r_0^2}{6\pi\epsilon_0 c^3}.
\end{aligned} \tag{37}$$

We see that the total radiated power does not depend on time: this is because we can see the integral over the solid angle as an average over all directions and in this circular uniform movement, averaging over φ is equivalent to time-averaging.

c) The angular distribution of the radiation is proportional to $\sin^2\alpha(t) = \left(\cos^2\theta + \sin^2(\omega_s t' - \varphi)\sin^2\theta\right)$ where $\alpha(t)$ is the angle between $\mathbf{a}(t')$ and $\mathbf{r}$. There is no radiation when $\mathbf{r}$ is parallel to $\mathbf{a}(t)$ and the radiation is maximum for $\mathbf{r}$ perpendicular to $\mathbf{a}(t)$.

The time average of the radiation is maximum in a direction always perpendicular to $\mathbf{a}(t)$, this is the case in the directions $\theta = 0$ and $\theta = \pi$, i.e. in the direction z of the external magnetic field. More generally, we can see the angular dependence $1+\cos^2\theta$ of the emitted power in equation (36).

Note on coordinate systems In principle the problems can be solved in any system of coordinates. However, in radiation problems, spherical coordinates (r, θ, φ) are usually more adapted because we take the large r limit and we often calculate the flux of the Poynting vector through a sphere.

However in order to take the scalar product and cross product of vectors, a fixed orthonormal triplet $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is more convenient than the triplet $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ which depend on θ and φ.

This is why in this exercise we use the parametrization $\mathbf{r} = r \sin\theta \cos\varphi \mathbf{e}_x + r \sin\theta \sin\varphi \mathbf{e}_y + r \cos\theta \mathbf{e}_z$ in the calculation (32) before switching back to $\mathbf{r} = r\mathbf{e}_r$.

3. *Classical atom*

Consider the classical model of the hydrogen atom:

- The proton, of charge $e = 1.60 \times 10^{-19}$ C and mass $m_p = 1.67 \times 10^{-27}$ kg, is at rest at the center of the atom.
- The electron, of charge $-e$ and mass $m_e = 9.11 \times 10^{-31}$ kg $\ll m_p$, moves around the proton in a circular orbit of radius $r_0 = 5.29 \times 10^{-11}$ m.

a) Calculate the frequency ν of this rotation.

b) Calculate the total power radiated by the system. Recall the formula

$$\mathbf{S}(t) = \frac{e^2}{16\pi^2\varepsilon_0 c^3} \frac{|\mathbf{a}|^2 \sin^2\alpha(t)}{r^2} \mathbf{e}_r \tag{38}$$

for the Poynting vector of a non-relativistic electron. Here, $\alpha(t)$ is the angle between the acceleration vector $\mathbf{a}$ of the electron and the observation direction $\mathbf{e}_r$.

c) Estimate the life time of the classical atom. Why are you still alive?

Solution

a) We are considering the electron as moving in a circular orbit around a fixed proton. Newton's law gives us immediately:

$$m_e \omega^2 r_0 = \frac{e^2}{4\pi\varepsilon_0 r_0^2}\,, \tag{39}$$

from which we can deduce the frequency:

$$\nu = \frac{\omega}{2\pi} = \sqrt{\frac{e^2}{16\pi^3 \epsilon_0 m_e r_0^3}} = 6.57 \times 10^{15}\,\mathrm{Hz}\,. \tag{40}$$

b) We have already computed the Poynting vector for an electron in a circular motion. It is:

$$\mathbf{S}(t) = \frac{e^2}{16\pi^2\varepsilon_0 c^3} \frac{|\mathbf{a}|^2 \sin^2\alpha(t)}{r^2} \mathbf{e}_r\,. \tag{41}$$

Notice that the velocity of the electron is $v = \omega r_0 = 2.2 \times 10^6\,\mathrm{m} \cdot \mathrm{s}^{-1} \ll c$ so our non-relativistic approximation is valid.

We can now compute the radiated power by integrating the flux of the Poynting vector over a sphere of radius R ($R \gg r_0$):

$$\begin{aligned} P = &\int_{S_R} R^2 \mathbf{S}(t) \cdot \mathbf{e}_r d\Omega = \frac{e^2}{16\pi^2 \varepsilon_0 c^3} \frac{r_0^2 \omega^4}{R^2} \int_{S_R} R^2 \sin^2 \alpha(t) d\Omega \\ =&\frac{e^2 r_0^2 \omega^4}{16\pi^2 \varepsilon_0 c^3} \int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\varphi (\cos^2\theta + \sin^2\theta \sin^2(\omega t' - \varphi)) \\ =&\frac{e^2 r_0^2 \omega^4}{16\pi^2 \varepsilon_0 c^3} \int_{-1}^{1} d\cos\theta [2\pi \cos^2\theta + \pi(1 - \cos^2\theta)] \\ =&\frac{e^2 r_0^2 \omega^4}{6\pi \varepsilon_0 c^3} = \frac{e^6}{96\pi^3 \varepsilon_0^3 c^3 m_e^2 r_0^4} \end{aligned} \tag{42}$$

We find the same result as in exercise 1 equation (37): the power radiated in a certain diretion depends on time but the total energy does not. This is due to the circular unifor movement of the electron.

c) An electron in a circular motion radiates energy so we expect the electron to spiral down towards the proton to compensate for the energy lost in radiation. While the problem is very complicated, we can treat it with the following approximations:

- At each time t we can consider the electron to be in a circular motion around the proton of radius $r(t)$, i.e. the electron falls slowly towards the proton.
- At each time t we can approximate the radiated power to be given by the result of question **b)**.

At each time t the energy of the system is:

$$E(t) = \frac{m_e}{2} r(t)^2 \omega(t)^2 - \frac{e^2}{4\pi\varepsilon_0 r(t)} = -\frac{e^2}{8\pi\varepsilon_0 r(t)}\,. \tag{43}$$

Then we can write:

$$\begin{aligned} \frac{dr}{dt} =&\frac{dr}{dE}\frac{dE}{dt} = \left(\frac{dE}{dr}\right)^{-1} (-P) = -\frac{8\pi\varepsilon_0 r^2}{e^2} \frac{e^6}{96\pi^3 \varepsilon_0^3 c^3 m_e^2 r^4} \\ =&-\frac{e^4}{12\pi^2 \varepsilon_0^2 c^3 m_e^2 r^2} \equiv -\frac{\alpha}{r^2}\,. \end{aligned} \tag{44}$$

This is a separable differential equation.

$$\begin{aligned} -r^2 dr =&\alpha dt \\ \left[-\frac{r^3}{3}\right]_{r_0}^{0} =&\alpha\tau \\ \tau =&\frac{r_0^3}{3\alpha} = \frac{4\pi^2 \varepsilon_0^2 c^3 m_e^2 r_0^3}{e^4}\,. \end{aligned} \tag{45}$$

Numerically, we have $\tau = 1.6 \times 10^{-11}$ s so it seems that the hydrogen atom is highly unstable.

The stability of atoms and thus of the world we know cannot be explained by classical electrodynamics. One needs to treat the problem in quantum mechanics. In this framework, one can show that the hydrogen atom has discrete energy levels with a lowest energy level at $E = -13.6$ eV which explains why the atom is stable.

Physically, our result (45) does not make sense and should only be taken as an order of magnitude because our approximations break down well before the radius of the trajectory goes to zero.

- First the non-relativistic assumption becomes wrong as $v \approx c$. In our circular trajectory appriximation, this is when

$$v = \omega r = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}} \approx c \implies r \approx \frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx 3 \times 10^{-15} \text{ m.} \tag{46}$$

- Second, our assumption that at any moment the trajectory can be considered circular. This is true if the relative rate of change of the radius $\frac{1}{r}\frac{dr}{dt}$ is small compared to the frequency ν (or ω). This breaks down when

$$\begin{aligned} \left|\frac{1}{r}\frac{dr}{dt}\right| = \frac{e^4}{12\pi^2\varepsilon_0^2 c^3 m_e^2 r^3} \approx \nu = \sqrt{\frac{e^2}{16\pi^3\epsilon_0 m_e r_0^3}} \\ \implies r \approx \frac{e^2}{\sqrt[3]{9\pi}\epsilon_0 m_e c^2} \approx 1 \times 10^{-14} \text{ m.} \end{aligned} \tag{47}$$

In any case there is still a large range of values $r = 10^{-11} - 10^{-14}$ m where our approximations are valid so it does not change our conclusion: the hydrogen atom seen classically is not stable because of energy loss by radiation.