Classical Electrodynamics

Week 9

1. Consider an infinite cylinder of radius a carrying the current I. The cylinder is surrounded by an insulator with magnetic permeability μ. A metallic cylindrical surface of radius $b > a$ conducts the current in the opposite direction.

Determine the magnetic field $\mathbf{H}$, the magnetic induction $\mathbf{B}$ and the magnetization $\mathbf{M}$ in every point in space. Find the free current density $\mathbf{J}$ and the average microscopic current density $\langle \mathbf{j} \rangle$.

2. In electrostatics, the n-th pole is given by

$$Q_{i_1 \ldots i_n} = \int d^3x\, \rho(\mathbf{x})\, T_{i_1 \ldots i_n}(\mathbf{x})\,, \tag{1}$$

where the totally symmetric tensor $T_{i_1 \ldots i_n}$ can be defined by

$$T_{i_1 \ldots i_n} = (2n-1)!!\; x_{i_1} \ldots x_{i_n} - A_{i_1 \ldots i_n}\,, \tag{2}$$

with the double factorial $(2n-1)!! = (2n-1)(2n-3)(2n-5)\ldots(5)(3)(1)$ and $(-1)!! = 1$. The tensor $A_{i_1 \ldots i_n}$ is an homogeneous polynomial of degree n in the components of $\mathbf{x}$ and it contains at least one Kronecker-δ so that the trace vanishes:

$$T_{i_1 \ldots i_n} \delta_{i_k i_l} = 0\,, \qquad \forall\, k, l \in \{1, 2, \ldots, n\}\,, \qquad k \neq l\,. \tag{3}$$

It is convenient to introduce the notation $B_{(i_1 \ldots i_n)}$ for the total symmetrization of a tensor $B_{i_1 \ldots i_n}$. More precisely,

$$B_{(i_1 \ldots i_n)} \equiv \frac{1}{n!} \sum_{perm\ \sigma} B_{\sigma(i_1 \ldots i_n)}\,, \tag{4}$$

where the sum runs over all permutations σ of the n indices $i_1 \ldots i_n$. For example,

$$B_{(ij)} \equiv \frac{1}{2}\left(B_{ij} + B_{ji}\right)\,, \qquad v_{(i} w_{j)} \equiv \frac{1}{2}\left(v_i w_j + v_j w_i\right) \tag{5}$$

$$B_{(ijk)} \equiv \frac{1}{6}\left(B_{ijk} + B_{ikj} + B_{jik} + B_{jki} + B_{kij} + B_{kji}\right)\,. \tag{6}$$

a) Argue that the tensors $A_{i_1 \ldots i_n}$, for $n = 2, 3, 4$, must be of the form

$$A_{ij} = c_2\, x^2 \delta_{ij} \tag{7}$$

$$A_{ijk} = c_3\, x^2 \delta_{(ij} x_{k)} \tag{8}$$

$$A_{ijkl} = c_4\, x^2 \delta_{(ij} x_k x_{l)} + c_4'\, x^4 \delta_{(ij} \delta_{kl)}\,, \tag{9}$$

where c_2, c_3, c_4 and c_4' are numerical constants.

b) Determine the constants c_2, c_3, c_4 and c'_4 imposing the trace condition (3).

c) How many independent components does the tensor $Q_{i_1 \dots i_n}$ have? Start by working out the cases $n = 0, 1, 2$. Can you guess the formula for general n? **Hint:** Start by counting the number of independent components in a totally symmetric tensor with n indices and then impose the trace constraint.

***d)** Generalize the previous question to d space dimensions. Show that the number of independent components of a traceless symmetric tensor $Q_{i_1 \dots i_n}$ in d dimensions (*i.e.*, each index can take the values $1, 2, \dots, d$) is given by

$$(2n + d - 2)\frac{(n + d - 3)!}{n!(d - 2)!}\,. \tag{10}$$

3. A dielectric sphere of radius a and permittivity ε_1 is placed in a constant electric field $\mathbf{E}_0$ in vacuum.

a) Assume that the electrostatic potential can be written as

$$\Phi = f(r) \cos\theta\,, \tag{11}$$

using spherical coordinates centred at the sphere and with the north pole direction $\theta = 0$ defined by the background electric field $\mathbf{E}_0$.

i. What is $f(r)$ in the absence of the sphere?

ii. What are the interface matching conditions for the electromagnetic fields $\mathbf{E}$ and $\mathbf{D}$ in the absence of free charges? What conditions do these imply for $f(r)$ at $r = a$?

iii. Determine the function $f(r)$ for all $r > 0$.
Hint: Firstly, list the conditions the function $f(r)$ satisfies at $r = 0$, $r = a$ and $r = \infty$. Secondly, derive differential equations for $f(r)$ in the regions $r < a$ and $r > a$. Thirdly, notice that $r\frac{d}{dr}r^l \propto r^l$.

b) Study the potential at large distances to show that the electric dipole of the sphere is given by

$$\mathbf{d} = \alpha\,\mathbf{E}_0\,, \tag{12}$$

and determine α.

c) Consider now a dilute gas of these small dielectric spheres. Let n be the number of spheres per unit volume. Determine the effective electric permittivity ε of this gas relevant to describe its electromagnetic properties at macroscopic length scales $\ell \gg n^{-\frac{1}{3}} \gg a$.