Classical Electrodynamics

Week 8

1. Consider the region $z > 0$ above an infinite conducting plane at $z = 0$.

a) There is a charge q at the position $\mathbf{x}_1 = h\,\mathbf{e}_z$ and a charge $-q$ at the position $\mathbf{x}_2 = \mathbf{x}_1 - a\,\mathbf{e}_x$. Find the electrostatic potential Φ in the region $z > 0$.

b) There is a dipole $\mathbf{d} = d\,\mathbf{e}_x$ at a distance h from the conducting plane.

i. Determine the electrostatic potential Φ in the region $z > 0$.
Hint: Use the previous question in the limit $a \to 0$ with $d = aq$ fixed.

ii. Show that at large distances the potential is dominated by a quadrupole and determine the corresponding quadrupole tensor Q_{ij}.

2. *Antenna*
A simple model of an antenna is given by the following current density:

$$\mathbf{J}(\mathbf{x}, t) = I\cos(\omega t)\Theta(a + z)\Theta(a - z)\delta(x)\delta(y)\mathbf{e}_z\,. \tag{1}$$

a) Use the continuity equation to calculate the charge density $\rho(\mathbf{x}, t)$, assuming the initial condition $\rho(\mathbf{x}, 0) = 0$. Verify that the total charge is conserved.

b) Calculate the total power radiated by the system. Recall that for this purpose it is sufficient to calculate the vector potential at very large distances $|\mathbf{x}| \gg \max(a, \lambda)$, where $\lambda = c/\omega$ is the wavelength of the emitted radiation. This is given by

$$\mathbf{A}(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\int d^3x'\,\mathbf{J}\left(\mathbf{x}', t - \frac{1}{c}|\mathbf{x}| + \frac{1}{c}\mathbf{n}\cdot\mathbf{x}'\right)\,, \qquad \mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|}\,. \tag{2}$$

Simplify your final result assuming that the source is *slow*: $\lambda \gg a$.

3. Consider the vacuum region $z > 0$. At the surface $z = 0$, the electrostatic potential is given, $\Phi(x, y, 0) = \phi_0(x, y)$. Assume that $\phi_0(x, y)$ goes to zero rapidly when $r = \sqrt{x^2 + y^2} \to \infty$, or equivalently, that it has compact support.

a) Find an integral expression for the potential Φ in terms of the boundary data ϕ_0.
Hint: Use an appropriate Green function as discussed in previous exercises.

b) Study the potential Φ for large values of $R = \sqrt{x^2 + y^2 + z^2}$. Show that the leading term in the large R expansion is the potential of a dipole. Write the dipole $\mathbf{d}$ in terms of ϕ_0.

c) Assuming that the dipole $\mathbf{d} = 0$, show that the leading term has the form of a quadrupole. Determine the quadrupole in terms of ϕ_0.

d) How do the these results change if we consider the potential ϕ in the region $z > 0$ and $y > 0$, with boundary values ϕ_0 given on the two semi-planes ($z = 0 \wedge y > 0$ and $y = 0 \wedge z > 0$) that bound the region. What is the leading multipole (in general) in this case?

4. We propose here a few questions to help you develop intuition about electrostatic multipoles.

a) Find a charge distribution $\rho_1(\mathbf{x})$ which has monopole q but dipole $\mathbf{d} = \mathbf{0}$. You can take it as simple as possible, but what follows works for any such charge distribution.

b) Consider the following charge distribution: $\rho_2(\mathbf{x}) = -\rho_1(\mathbf{x}) + \rho_1(\mathbf{x} - a\mathbf{e}_x)$. How does it look like ? Compute its monopole and dipole. Is the result surprising ?

c) Now consider the charge distribution $\rho_3(\mathbf{x}) = \rho_1(\mathbf{x}) + \rho_2(\mathbf{x})$. How does it look like ? Compute its monopole and dipole. Is the result surprising ?

d) Under what conditions are the monopole, the dipole or the quadrupole of a charge distribution invariant under translation of the charge density ? Can you generalize the result ?

e) Find a charge distribution which has zero monopole, zero dipole, and with only non-zero quadrupole components $Q_{12} = Q_{21} \neq 0$. Do the same with only $Q_{23} = Q_{32} \neq 0$, and also with only $Q_{13} = Q_{31} \neq 0$. Can you have only $Q_{11} \neq 0$? Find one with only $Q_{11} = -Q_{22} \neq 0$, one with only $Q_{22} = -Q_{33} \neq 0$ and finally one with only $Q_{11} = -Q_{33} \neq 0$.

f) If not already done, complete exercise **2.** of last week.