

Classical Electrodynamics

Week 7

1. Calculate the leading behaviour of the electrostatic potential $\Phi(\mathbf{x})$ at large distances $|\mathbf{x}| \gg a$ for the following charge distributions:
 - a) One charge q at the point $\mathbf{x}_0 = a \mathbf{e}_x$ and one charge q at the point $-\mathbf{x}_0$.
 - b) One charge q at the point $\mathbf{x}_0 = a \mathbf{e}_x$ and one charge $-q$ at the point $-\mathbf{x}_0$.
 - c) Two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_x + a \mathbf{e}_y$ and $\mathbf{x}_2 = a \mathbf{e}_x - a \mathbf{e}_y$ and two charges $-q$ at the points $-\mathbf{x}_1$ and $-\mathbf{x}_2$.
 - d) Two charges q at the points $\mathbf{x}_1 = a \mathbf{e}_x + a \mathbf{e}_y$ and $-\mathbf{x}_1$ and two charges $-q$ at the points $\mathbf{x}_2 = a \mathbf{e}_x - a \mathbf{e}_y$ and $-\mathbf{x}_2$.
 - e) Four charges q and four charges $-q$ placed at the eight vertices of a cube of side a , such that any two charges at distance a have opposite sign.
 - f) The following linear charge density along a ring of radius a :

$$\rho(r, \varphi, z) = \frac{q}{2\pi a} \cos(n\varphi) \delta(z) \delta(r - a), \quad (1)$$

where $n = 0, 1, 2, \dots$ and we used cylindrical coordinates (r, φ, z) .

Hint: The following identity may be useful, at small x we have

$$\begin{aligned} \frac{1}{\sqrt{1+x^2-2xy}} &= \sum_{k=0}^l P_k(y) x^k + O(x^{l+1}) \\ &= \sum_{k=0}^l \left[\frac{(2k-1)!!}{k!} y^k + O(y^{k-2}) \right] x^k + O(x^{l+1}), \end{aligned} \quad (2)$$

where $P_k(y)$ is the Legendre polynomial of degree k and we wrote explicitly the leading term in the last line.

2. Consider a particle of charge q moving with constant velocity $\mathbf{v} = (0, 0, v)$ along the z -axis. In a previous exercise, we have found that the retarded potentials are given by

$$\Phi(t, \mathbf{x}) = \frac{q\gamma}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}}, \quad \mathbf{A}(t, \mathbf{x}) = \frac{\mathbf{v}}{c^2} \Phi(t, \mathbf{x}), \quad (3)$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor.

- a) Starting from the potentials, determine the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

- b) Write the electromagnetic fields in terms of the norm of the vector $\mathbf{R}_t$ (vector between the charge q and the observer at time t) and the angle $\theta = \angle(\mathbf{R}_t, \mathbf{v})$. Study the fields in the non-relativistic ($v \ll c$) and ultra-relativistic ($v \approx c$) limits.

3. A linearly polarised electromagnetic wave travels along the z -axis. It propagates in vacuum for $z < 0$ until it reaches a conductor, with conductivity σ , that fills the space for $z > 0$. The electric field of the incident wave is given by

$$\mathbf{E}_{in} = \text{Re} [E_0 e^{i(kz - \omega t)}] \mathbf{e}_x \quad (4)$$

where $k = \omega/c$.

- a) Show that the local Ohm's law $\mathbf{J} = \sigma \mathbf{E}$ and Maxwell equations imply that

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} + \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (5)$$

inside the conductor. For simplicity, we have assumed that the conductor has the electric permittivity and magnetic permeability of the vacuum.

- b) Use the last equation to derive the form of the magnetic field inside the conductor,

$$\mathbf{B} = \text{Re} [A e^{i(k'z - \omega t)}] \mathbf{e}_y, \quad z > 0, \quad (6)$$

where A is an integration constant and $k' = \sqrt{k^2 + i\omega\mu_0\sigma}$.

- c) What is the form of the electric field inside the conductor? How deep inside the conductor does the wave penetrate?
- d) The electric field outside the conductor ($z < 0$) is a superposition of the incident and the reflected wave,

$$\mathbf{E} = \text{Re} [E_0 e^{ikz - i\omega t} + E_1 e^{-ikz - i\omega t}] \mathbf{e}_x, \quad z < 0. \quad (7)$$

Determine E_1 and A imposing continuity of the electric field and its first derivative at the interface $z = 0$. Notice that these follow from Maxwell equations in the absence of surface charge density and surface current. Show that the fraction of incident power reflected by the conductor is given by

$$\left| \frac{k' - k}{k' + k} \right|^2. \quad (8)$$

- e) Verify that the rest of the incident power is dissipated in the conductor. Recall that the power dissipated per unit volume is given by $\mathbf{J} \cdot \mathbf{E}$.
- f) Discuss your previous results in the high and low frequency limits ($\omega \gg \sigma/\epsilon_0$ and $\omega \ll \sigma/\epsilon_0$). What is the relevant regime for light incident over aluminium?