Classical Electrodynamics

Week 6

1. Consider a particle of charge q moving with constant velocity $\mathbf{v} = (0, 0, v)$ along the z-axis. Show that the retarded potentials are given by

$$\Phi(t, \mathbf{x}) = \frac{q\gamma}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}} \,, \qquad \mathbf{A}(t, \mathbf{x}) = \frac{\mathbf{v}}{c^2}\Phi(t, \mathbf{x}) \,, \tag{1}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor.

2. Two infinitely-long grounded metal plates, located at $y = 0$ and at $y = a$, are connected at $x = \pm b$ by metal strips (again infinitely-long) maintained at a constant potential V_0 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe, by using the separation of variable method.

Hint: Use translational symmetry to eliminate the z dependence, and then start by looking for solutions to the Poisson equation that have the form:

$$\Phi(x, y) = X(x)Y(y).$$

Work on decoupling the variables and then solve the equations you obtain: what important property does belong to the set of solutions you found?

3. The electric field associated with an electromagnetic wave travelling along the z-axis can be written as follows

$$\begin{aligned}\mathbf{E} &= E_{0x} \cos\left(kz - \omega t + \theta\right) \mathbf{e}_x + E_{0y} \cos\left(kz - \omega t + \theta + \phi\right) \mathbf{e}_y & (2)\\ &= \mathrm{Re}\left[\left(J_x\mathbf{e}_x + J_y\mathbf{e}_y\right)\sqrt{E_{0x}^2 + E_{0y}^2}\, e^{i(kz-\omega t+\theta)}\right] & (3)\end{aligned}$$

where the polarization of the wave is encoded in the two dimensional vector,

$$\mathbf{J} = (J_x, J_y) = \left(\frac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}}, \frac{E_{0y}}{\sqrt{E_{0x}^2 + E_{0y}^2}} e^{i\phi}\right) \,, \tag{4}$$

known as the Jones vector. Notice that $|\mathbf{J}|^2 = |J_x|^2 + |J_y|^2 = 1$.

Consider two electromagnetic waves $\mathbf{E}_1$ and $\mathbf{E}_2$ propagating along the z-axis with opposite circular polarisations and with the same frequency and phase.

a) Find the Jones vector of each wave $\mathbf{E}_1$ and $\mathbf{E}_2$.

b) Discuss, as a function of the amplitudes of each wave, the polarisation of the total wave and its associated Jones vector.

4. The complex plane is a very useful tool to compute integrals. The Cauchy theorem

$$\oint_{\partial D} f(z)dz = 0 \ , \qquad \text{if } f \text{ is analytic in } D \, , \tag{5}$$

tells us that we can continuously deform the integration contour without changing the value of the integral (as long as we keep the endpoints fixed and do not cross any singularity). The residue theorem

$$\oint_{\partial D} f(z)dz = 2\pi i \sum_k \mathrm{Res}(f, z_k) \, , \tag{6}$$

reduces the contour integral calculation to the sum over residues of all poles of f inside the domain D (as long as f is single-valued inside D).

a) Use these ideas to calculate the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} \ , \qquad \int_{-\infty}^{\infty} \frac{dx}{1+x^4} \ . \tag{7}$$

b) Calculate also the Fourier transforms

$$\int \frac{d\omega}{2\pi} \frac{1}{1+\omega^2} e^{-i\omega t} \, , \qquad \int \frac{d\omega}{2\pi} \frac{1}{\cosh \frac{\pi\omega}{2}} e^{-i\omega t} \, . \tag{8}$$

c) Use the same methods to find the form of the advanced and retarded Green functions in position space, starting from

$$G(\mathbf{x}, t) = \int \frac{d\omega d^3k}{(2\pi)^4} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\mathbf{k}^2 - \frac{1}{c^2}(\omega \pm i\epsilon)^2} \, , \tag{9}$$

where $\epsilon > 0$ is infinitesimal and the sign $\pm$ distinguishes the retarded from the advanced Green function.