Classical Electrodynamics

Week 5

1. Consider the interior $\mathcal{V}$ of a spherical shell with radius R centred at point $\mathbf{x}_0$. There is no electric charge inside the shell and there is an arbitrary profile of electrostatic potential on the shell $\Phi(\mathbf{x})\big|_{|\mathbf{x}|=R} = V(\theta, \varphi)$.

a) Using the Green's identity for two scalar Φ and Ψ:

$$\int_{\mathcal{V}} d^3r \, (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) = \oint_{\partial \mathcal{V}} \mathbf{dS} \cdot (\Phi \nabla \Psi - \Psi \nabla \Phi), \tag{1}$$

show that the potential at center $\mathbf{x}_0$ is the average of potential $V(\theta, \varphi)$ over the shell:

$$\Phi(\mathbf{x}_0) = \int_{\partial \mathcal{V}} V(\theta, \varphi) \, d\Omega \,, \tag{2}$$

with $d\Omega = \frac{1}{4\pi} \sin\theta \, d\theta \, d\varphi$.

Hint 1: Introduce a function Ψ that simplifies the l.h.s of equation (1) to the potential at center i.e. $\Phi(\mathbf{x}_0)$.

Hint 2: Use $\oint_{\partial \mathcal{V}} \mathbf{dS} \cdot \nabla \Phi(\mathbf{x}) = 0$ to simplify the r.h.s of equation (1).

b) Use this result to argue that maxima or minima of a solution to Laplace equation can only be at the boundary of the domain.

2. Consider a sphere of radius R with a fixed potential $\Phi(R, \theta, \varphi) = V(\theta, \varphi)$ on its surface.

a) Find an integral expression for the potential $\Phi(r, \theta, \varphi)$ in all space outside the sphere. **Hint:** Start by finding the appropriate Green function $G(\mathbf{r}, \mathbf{r}')$ for this problem. Recall the general solution of Poisson equation

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}', \mathbf{r}) d^3\mathbf{r}' + \int_{\partial V} [G(\mathbf{r}', \mathbf{r}) \nabla_{\mathbf{r}'} \Phi(\mathbf{r}') - \Phi(\mathbf{r}') \nabla_{\mathbf{r}'} G(\mathbf{r}', \mathbf{r})] \cdot \mathbf{d}\sigma' \,.$$

b) What is the leading behaviour of the potential $\Phi(r, \theta, \varphi)$ far away from the sphere, *i.e.* for $r \gg R$?

c) Consider now the following experiment. We take a metallic ball, cut it in half and glue it back together using an insulating glue. Then we establish a potential difference V_0 between the two half-balls (keeping the total charge of the ball equal to zero). Find an integral expression for the potential Φ outside the ball. How does Φ behave far away from the ball?

d) Estimate the total charge accumulated in each half-ball assuming that the thickness d of the insulating glue is much smaller than the radius of the ball.

e) Assume that the interior of the sphere is empty and the potential on the surface is given by $\Phi(R, \theta, \varphi) = V(\theta, \varphi)$. Find an integral expression for the potential $\Phi(r, \theta, \varphi)$ inside the sphere. What is the potential at the center of the sphere?

3. Consider the general expressions for the potentials Φ and $\mathbf{A}$ using the retarded Green function,

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \, \frac{\rho\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x} - \mathbf{x}'|\right)}{|\mathbf{x} - \mathbf{x}'|} \,, \tag{3}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\mathbf{J}\left(\mathbf{x}', t' = t - \frac{1}{c}|\mathbf{x} - \mathbf{x}'|\right)}{|\mathbf{x} - \mathbf{x}'|} \,. \tag{4}$$

Show that the Lorenz gauge condition is implied by the continuity equation, *i.e.* charge conservation.

Hint: Recall that the continuity equation reads

$$\frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \nabla_{\mathbf{x}} \cdot \mathbf{J}(\mathbf{x}, t) = 0 \,, \tag{5}$$

and the $\nabla_{\mathbf{x}}$ should be understood as partial derivative on the spatial component. You will also need the property

$$(\nabla_{\mathbf{x}'} + \nabla_{\mathbf{x}}) f(\mathbf{x} - \mathbf{x}') = 0 \ . \tag{6}$$

4. The Faraday disk is a simple electrical generator. The basic ingredients are depicted in figure 1. There is a metallic annulus with internal radius r_1, external radius r_2 and width h. The annulus is placed in a region with a constant magnetic field perpendicular to the plane of the annulus. The metal of the annulus has electrical conductivity σ.

a) When the annulus is rotating with angular velocity ω and the external circuit is open, what is the electrostatic potential difference V_0 between the internal and the external circumference of the annulus?

b) This potential difference can be used as an electrical generator if we connect the internal and the external circumference of the annulus to a circuit as shown in figure 1 (assume that the connection to the external circuit does not break the cylindrical symmetry of the system). What is the potential difference V (between r_1 and r_2) when there is a current I flowing through the circuit?
Hint 1: Recall the microscopic definition of the conductivity σ,

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \ . \tag{7}$$

Hint 2: It might be useful to determine the resistance of the annulus.

c) What is the maximum output of electrical power of this generator? What is the efficiency of the generator when working at the maximal output power?

d) Assume that the metallic annulus is mechanically connected to the wheels of a train so that they rotate with the same angular velocity ω. Assume also that the external electrical circuit acts as an effective resistance R. In this case, we can use the annulus as a regenerative braking system of the train. Start by showing that the total kinetic energy of the train can be written as $E = \frac{1}{2} I_{eff} \omega^2$ and estimate the effective moment of inertia I_{eff}. Then, using conservation of energy, find an equation for the time evolution of the angular velocity ω when the brakes are on, *i.e.* the circuit is closed. How long does it take to reduce the velocity of the train by a factor of 2?

e) **Challenge**: Suppose the annulus is made of $300\,kg$ of iron, the train weighs $400\,ton$ and it is travelling at $200\,km/h$. If we brake the train by short-circuiting the internal and the external circumference of the annulus, will the annulus melt? Is the braking event safe for the passengers?

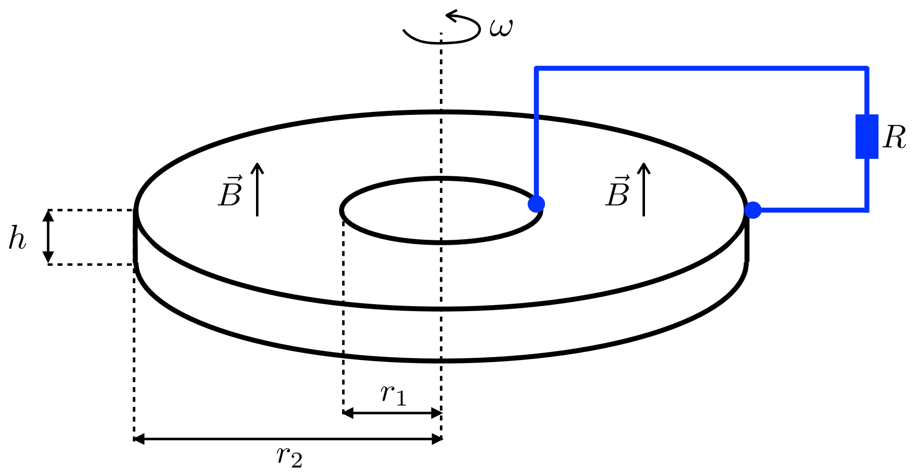

Figure 1: Metallic annulus rotating in a constant magnetic field **B**.