Classical Electrodynamics

Week 4

1. Two spherical cavities, of radii a and b, are hollowed out from the interior of a neutral conducting sphere of radius R. At the center of each cavity a point charge is placed. Call these charges q_a and q_b.

a) Find the surface charges σ_a, σ_b and σ_R.

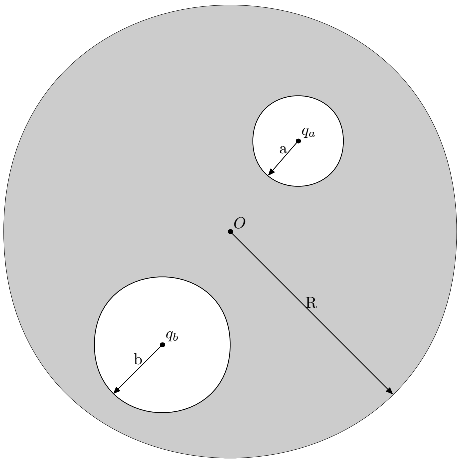
b) What is the field outside the conductor?

c) What is the field within each cavity?

d) What is the force on q_a and q_b?

e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

Hint: it is possible to solve this exercise without any computation beyond elementary algebra.



2. Given the charge density $\rho(x, y, z)$ and the value of the potential $\Phi(x, y, 0) = \varphi(x, y)$ on the plane $z = 0$, determine the potential $\Phi(x, y, z)$ in the region $z \geq 0$.

Hint: Start by finding the appropriate Green function $G(\mathbf{r}, \mathbf{r}')$ for this problem. Recall the general solution of Poisson equation

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r}')G(\mathbf{r}', \mathbf{r})d^3\mathbf{r}' + \int_{\partial V} [G(\mathbf{r}', \mathbf{r})\nabla\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla_{\mathbf{r}'}G(\mathbf{r}', \mathbf{r})] \cdot \mathbf{d}\sigma' .$$

3. Consider the region $\rho \leq \rho_0$ and $0 \leq \varphi \leq \beta$ in cylindrical coordinates as depicted in figure 1. There are no charges inside this region and the potential is fixed at

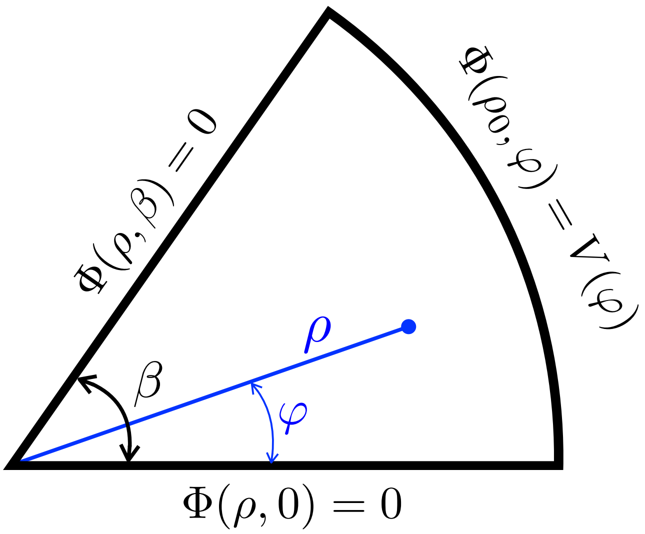


Figure 1: Two dimensional cross section of a region that extends along the z-direction orthogonal to the plane of the figure. The system is invariant under translations in the z-direction. The radial conductors at $\varphi = 0$ and $\varphi = \beta$ are grounded. The surface at $\rho = \rho_0$ has a given potential profile $V(\varphi)$. The angular region bounded by these 3 surfaces is empty (vacuum).

the boundaries to $\Phi(\rho, \varphi = 0, z) = \Phi(\rho, \varphi = \beta, z) = 0$ and $\Phi(\rho_0, \varphi, z) = V(\varphi)$, where $V(\varphi)$ is a continuous function for $0 \leq \varphi \leq \beta$ and $V(0) = V(\beta) = 0$.

a) Check that the potential inside the region is given by

$$\Phi(\rho, \varphi, z) = \sum_{m=1}^{\infty} a_m \left(\frac{\rho}{\rho_0}\right)^{m\pi/\beta} \sin\frac{m\pi\varphi}{\beta} \tag{1}$$

and determine the coefficients a_m in terms of V.

b) Determine the surface charge density $\sigma(\rho)$ along the boundary $\varphi = 0$ of the grounded conductor.

c) Assuming the generic case $a_1 \neq 0$, study the behaviour of the surface charge density $\sigma(\rho)$ near the corner at $\rho = 0$. Comment on the 4 cases $0 < \beta < \pi$, $\beta = \pi$, $\pi < \beta < 2\pi$ and $\beta = 2\pi$ and their physical meaning.

d) **Challenge**: Take $\beta = 3\pi/2$ and suppose the region is filled with air. What will happen near the corner at $\rho = 0$?