Classical Electrodynamics

Week 2

1. Show that

a) $\delta_{ii} = 3$

b) $\delta_{ij}\epsilon_{ijk} = 0$

c) $\epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij}$

d) $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

2. By looking at a component and without writing the vectors explicitly, prove the following identities:

a) $\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

b) $\vec{\nabla} \cdot \left(\vec{A} \wedge \vec{B}\right) = \left(\vec{\nabla} \wedge \vec{A}\right) \cdot \vec{B} - \vec{A} \cdot \left(\vec{\nabla} \wedge \vec{B}\right)$

c) $\vec{\nabla} \wedge \left(\vec{A} \wedge \vec{B}\right) = \vec{A}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{B}\left(\vec{\nabla} \cdot \vec{A}\right) + \left(B \cdot \vec{\nabla}\right)\vec{A} - \left(A \cdot \vec{\nabla}\right)\vec{B}$

d) $\left(\vec{A} \wedge \vec{\nabla}\right) \cdot \vec{B} = \vec{A} \cdot \left(\vec{\nabla} \wedge \vec{B}\right)$

3. Gauge transformations

a) Show that the potentials

$$\phi = 0 \qquad \mathbf{A} = \frac{B}{2}(-y, x, 0) \tag{1}$$

$$\phi' = 0 \qquad \mathbf{A}' = B(0, x, 0) \tag{2}$$

are equivalent and represent the same magnetic and electric fields. Find the gauge transformation that relates them. Here B is a constant and (x_1, x_2, x_3) represents the explicit components of a vector.

b) Show that the potentials

$$\phi = -\mathbf{E}_0 \cdot \mathbf{r}\sin(\omega t) \qquad \mathbf{A} = \mathbf{0} \tag{3}$$

$$\phi' = 0 \qquad \mathbf{A}' = \mathbf{E}_0 \frac{1}{\omega}\cos(\omega t) \tag{4}$$

are equivalent and represent the same magnetic and electric fields. Find the gauge transformation that relates them.

4. Gradient in spherical coordinates Starting from the expression of the gradient in cartesian coordinates

$$\left(\vec{\nabla} f\right)_{\text{cartesian}} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

derive the expression for the gradient in spherical coordinates

$$\left(\vec{\nabla} f\right)_{\text{spherical}} = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right).$$

5. Dirac δ-function

a) Show that

$$\lim_{\alpha\to 0} \frac{\alpha}{\pi(\alpha^2 + x^2)} \tag{5}$$

with $\alpha > 0$ is a representation of the δ-function by verifying that $\int_{-\infty}^{\infty} dx\delta(x)f(x) = f(0)$, where f is a smooth test function that does not grow at infinity.

b) Prove the following identity

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)\ , \tag{6}$$

where the sum is over all the zeros $f(x_i) = 0$ and we assume that $f'(x_i) \neq 0$.

c) Prove the following identity in $\mathbb{R}^n$

$$\int d^n\mathbf{x} f(\mathbf{x})\nabla_{\mathbf{x}}\delta^n(\mathbf{x} - \mathbf{x}_0) = -\left.\nabla_{\mathbf{x}} f\right|_{\mathbf{x}=\mathbf{x}_0}\ . \tag{7}$$

d) Let $g(x)$ be a bounded smooth function. Compute $\int dx g(x)\theta'(x - x_0)$ and derive a relation between the Heaviside θ-function and the Dirac δ-function.

e) Evaluate the following integrals

$$\int dx g(x)\delta'(x - x_0)\ , \qquad\qquad \int dx g(x)\delta''(x - x_0)\ . \tag{8}$$

f) Calculate $\exp\left(x_0\frac{d}{dx}\right)\delta(x)$.

g) Show that $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}$.