Classical Electrodynamics

Week 14

1. Consider a particle of mass m and charge e in a constant electromagnetic field $\mathbf{E} = E\mathbf{e}_y$, $\mathbf{B} = B\mathbf{e}_z$ with $B > E/c$.

   (a) Find the trajectory of the particle in the non-relativistic case.

   (b) Find the trajectory of the particle in the relativistic case using a boost to move to a reference frame where the electric field vanishes.

2. *Relativistic Larmor formula*
   Calculate the energy radiated per unit time $\frac{d\mathcal{E}}{dt}$ by a charged particle as a function of its velocity and the background electromagnetic fields. For simplicity, analyse the following cases:

   (a) $\mathbf{v} \parallel \mathbf{E}$

   (b) $\mathbf{v} \parallel \mathbf{B}$

   (c) $\mathbf{v} \parallel \mathbf{E} \times \mathbf{B}$

   (d) In the case of circular accelerator (synchrotron), with $\mathbf{E} = 0$, show that the energy loss is proportional to $\mathcal{E}^4$.
   Hint: Use the Larmor radius $\rho = \frac{m\gamma v}{qB}$ and the relativistic approximation $v \sim c$.

3. *Kramers-Kronig relations for refractive index.*

   In this exercise we will review a few steps in the derivation of Kramers Kronig relations, and derive the relation between real and imaginary parts of the refractive index $\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$ based on the analytic properties of electric permittivity $\varepsilon(\omega)$.

   (a) As a first step, we want to study the behavior of $\varepsilon(\omega)$ for real and very large frequencies. You can derive that such behavior is universal, regardless of whether the material is a dielectric or a conductor.
   Compute the polarization $\vec{P}$ of the body when the frequency of the incident (monochromatic) electromagnetic wave is much larger than the motion frequency of any atomic electron, and derive the correspondent electric permittivity in this regime.
   Hint: Motivate why at very high frequencies electric field can be thought of being uniform in space when determining electrons acceleration from the field.

   (b) Argue that for $\omega \to \infty$ in any direction in the upper-half plane $\varepsilon(\omega) \to \epsilon_0$.

   (c) Show that $\varepsilon(\omega)$ cannot acquire any real value on the upper half plane, except on the imaginary axes, where it is monotonically decreasing. From here, conclude that $\varepsilon(\omega) \neq 0$ in the upper-half ω-plane.

Hint 1: It can be heplful to recall the theorem for any function g of complex variables z, which states that the integral

$$\frac{1}{2i\pi}\oint_{\mathcal{C}} \frac{\mathrm{d}g(z)}{\mathrm{d}z}\frac{\mathrm{d}z}{g(z)-a} = \#_{\text{zeros}} - \#_{\text{poles}}, \tag{1}$$

where a is any constant, $\#_{\text{zeros}}$ is the number of zeros of the function $g(z)-a$ within the contour $\mathcal{C}$, and $\#_{\text{poles}}$ is the number of poles of $g(z)-a$ inside $\mathcal{C}$.

Hint 2: The function $\varepsilon(\omega)$ always has a non-vanishing imaginary part when ω is real and different from 0. This is due to the dissipation any electromagnetic wave is subjected to when traveling through a medium.

(d) Assuming constant magnetic permeability, so that $\tilde{n}(\omega) = \sqrt{\varepsilon(\omega)\mu}$, derive the Kramers-Kronig relations for $\tilde{n}(\omega)$. Think about a good strategy for measuring the refractive index, $\tilde{n}(\omega)$, given the derived relations.

Bonus questions

These are bonus questions. They are redundant with exercises already seen in class but can be used as extra exercises to practice.

1. *Lorentz transformation of electromagnetic fields III*
   Consider a constant electromagnetic field $\{\mathbf{E}, \mathbf{B}\}$ such that $\mathbf{E}\cdot\mathbf{B} = 0$ in the reference frame $\mathcal{R}$.

   (a) Find a reference frame $\mathcal{R}'$ where either $\mathbf{E}' = 0$ or $\mathbf{B}' = 0$.

   (b) Is this always possible? Is the solution unique?

   (c) Compute the magnitudes of $\mathbf{E}'$ and $\mathbf{B}'$ in the reference frame $\mathcal{R}'$.