Classical Electrodynamics

Week 13

1. *Relativistic Doppler effect*
An electromagnetic wave is described by the fields:

$$\mathbf{E} = \mathbf{E}_0 \sin(\omega t - \mathbf{k}\cdot\mathbf{x}) \ , \qquad \mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E} \ , \qquad \mathbf{E}_0\cdot\mathbf{k} = 0 \ , \tag{1}$$

where ω is the frequency in the laboratory reference frame and $\mathbf{k} = \omega/c\,\mathbf{e}_z$ is wave vector. An observer moves at constant velocity v along the z-axis. Find the frequency ω' of the wave in the reference frame of the moving observer. Show that $\left(\frac{\omega}{c}, \mathbf{k}\right)$ is a four-vector.
Hint: The phase of an electromagnetic wave is Lorentz invariant.

2. A conducting loop with a rectangular shape of sides a' and b' supports the current I' (in the reference frame where the loop is at rest). The cross section of the wire of the loop is S'. The loop moves at constant speed v in the direction parallel to the side of length a'.
Find the charge distribution and currents in each side of the loop in the reference frame of the laboratory. Comment on your results.

3. *The Breakthrough Starshot program*
The stress tensor associated to electromagnetic fields is

$$T^{\mu\nu} = \frac{1}{\mu_0 c^2}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right),$$

where $F^{\mu\nu}$ is the field-strength tensor.

Remark: The questions are formulated using the $(-,+++)$ convention for the Minkowski metric.

(a) Show that, in terms of the electromagnetic fields, the components of $T^{\mu\nu}$ reduce to:

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & \vec{S}/c \\ \vec{S}/c & -\tau_{ij} \end{pmatrix},$$

where $\mathcal{E}$ is the electromagnetic energy density, $\vec{S}$ is the Poynting vector and τ_{ij} are the components of the Maxwell stress tensor, defined as:

$$\tau_{ij} := \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right).$$

(b) Compute the divergence $\partial_\nu T^{\mu\nu} = -f^\mu$ and show that f^μ is a 4−vector. What physical quantities do its components represent? **Hint:** use Maxwell equations in presence of charges.

The Breakthrough Starshot program aims at developing an ultra-fast spaceship for interstellar missions. The prototype is made of a very light nanocraft attached to a lightsail, propelled by a ground-based light beamer. You can find a sketch of it in picture 1.

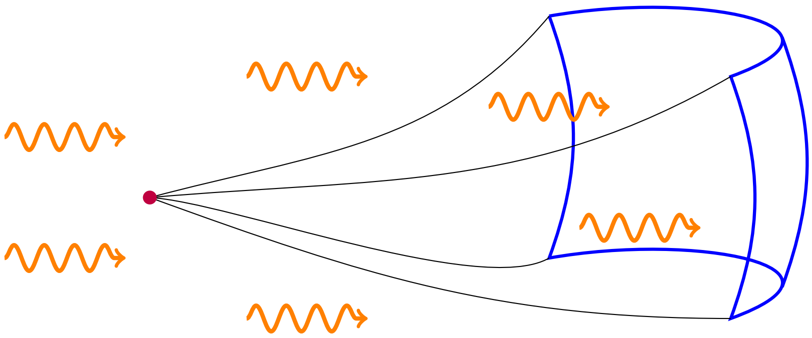

Figure 1: Sketch of a prototype of the Breakthrough Starshot program. In blue is depicted the lightsail, the electromagnetic wave arrives perpendicularly, to which is attached the purple nanocraft.

(c) Assuming the incident electromagnetic wave is perfectly reflected by the light sail (and the light sail is flat), compute the power per unit area w transferred to the light sail by an incident electromagnetic wave of amplitude $|\vec{E}| = c|\vec{B}|$ (in the reference frame where the light sail is at rest).

(d) The project aims at pushing the nanocraft to travel at speed $c/5$. What is the ratio between the power w' which needs to be injected in the light beam from Earth, and the power w transferred to the light sail in its reference frame?