Classical Electrodynamics

Week 12

1. *Symmetries of the wave equation – Lorentz Boost*
During the lecture, you derived the symmetries of the wave equation using *Wick* rotation. Here, you will rederive them independently. For simplicity, we will consider only one spacial dimension to focus on the symmetry involving space and time.

The d'Alembertian operator can be written in matrix notation

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\eta} \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix}, \tag{1}$$

and the most general linear transformation of spacetime is

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \end{pmatrix} \tag{2}$$

a) Suppose that $\Psi(x,t)$ is a solution of $\Box\Psi(x,t) = 0$. What if the constraints on Λ such that $\Psi'(x,t) = \Psi(x',t')$ is also a solution?

b) Show that the constraints imply that

$$\Lambda = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \tag{3}$$

with χ a free parameter.

c) By studying the trajectory of the origin $x' = 0$, relate the rapidity χ to $\beta = v/c$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, where v is the velocity of $\mathcal{O}'$.

d) Write Λ in term of β, γ. Do you recognize the familiar Lorentz boost?

2. *Invariance of Electric potential*
Consider a scalar potential ϕ.

a) In electrostatics in the absence of charges, the scalar potential ϕ satisfies the Laplace equation $\Delta\phi = 0$. Find the set of transformations of space that leave this equation invariant.

b) In electrodynamics in Lorenz gauge, the scalar potential satisfies the wave equation $\Box\phi = 0$. Find the set of transformations of space and time that leave this equation invariant.
Note: This exercise is similar in spirit as 1a). Here try to derive to work in tensor notation and in $3+1$ dimensions.

3. Determine the potentials $(\phi, \mathbf{A})$ of a charge q at rest at the origin of a reference frame $\mathcal{R}$. Then consider a reference frame $\mathcal{R}'$ moving with uniform velocity $\mathbf{v}$ with respect to $\mathcal{R}$. What are the electromagnetic potentials $(\phi', \mathbf{A}')$ in the reference frame $\mathcal{R}'$? Compare your results with the Liénard-Wiechert potentials.

4. The predictions of special relativity are often counter-intuitive because our daily experience is limited to velocities much smaller than the speed of light. The goal of this exercise is to develop your relativistic intuition through the discussion of several thought experiments.

Suggestion: You are not expected to do long computation. Reflect on the situation and draw a spacetime diagram describing each experiment.

a) Two identical rockets are at rest connected by a rope of length 100 m. The rope is stretched and cannot be extended beyond 101 m without breaking. The two rockets are programmed to start accelerating exactly at the same time along the direction parallel to the rope. They stop accelerating when they reach the velocity 0.8c. Did the rope break?

b) John wants to put a 10 m long ladder inside his garage, which also has length 10 m. The garage has doors at both ends that can be closed simultaneously by pressing a button in the middle of the garage. Paul tells John to run with the ladder towards the garage while he stays next to the button. Since John can run at the speed $0.8c$ the ladder will be Lorentz contracted and Paul will easily be able to close the garage doors with the ladder inside. On the other hand, for John it is the garage that gets Lorentz contracted and therefore it will not be possible to close the garage doors. Who is right?

c) Consider two LED's and two photo-detectors placed at the four vertices of a vertical square of side 10 m. The LEDs are placed in the bottom vertices of the square and the photo-detectors on the top vertices. Each LED emits light towards the photo-detector just above it (on the the same side of the square) and the two photo-detectors are connected to an electronic NOR that will ring an alarm if and only if none of the photo-detectors detects light at the same time. Now consider an opaque bar of length 10 m that crosses the square horizontally at the speed $c/2$ (and at mid height). From the point of view of the square the bar gets Lorentz contracted and will not be able to block the light of both LEDs at the same time. However, from the point of view of the bar, the square gets Lorentz contracted and both LEDs will be blocked at the same time during some time. Will the alarm ring?