Classical Electrodynamics

Week 11

Note: Exercise **0.** marked by (*) is optional and is similar to exercise **1.** of week 2. We suggest that you start with exercise **1.** We left it there as it provide a good exercise to practice the tensor notation and the formula can be useful for the rest of the problem sheet.

We use the metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$.

0. (*) *Lévi-Cività symbol*

The Lévi-Cività symbol is defined by

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu\nu\rho\sigma) \text{ is an even permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is an odd permutation of } (0123) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

a) Defining the Lorentz transformation of $\varepsilon^{\mu\nu\rho\sigma}$ by

$$\varepsilon'^{\mu\nu\rho\sigma} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\Lambda^{\rho}{}_{\gamma}\Lambda^{\sigma}{}_{\delta}\varepsilon^{\alpha\beta\gamma\delta}\,, \tag{2}$$

show that $\varepsilon'^{\mu\nu\rho\sigma} = \det(\Lambda)\varepsilon^{\mu\nu\rho\sigma}$ and that $\det(\Lambda) = \pm 1$.

b) Verify the following identities

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -4! \tag{3}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = -6\,\delta^{\mu}_{\alpha} \tag{4}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \tag{5}$$

1. *Maxwell equations*

Using the definition of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A^\mu = (\Phi, c\mathbf{A})^T$, verify that the equations

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\varepsilon_0} j^\nu \tag{6}$$

$$\partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0 \tag{7}$$

describe Maxwell equations. Show that equations (6) and (7) are Lorentz invariant.

2. *Lorentz invariants*

a) Using the field strength $F_{\mu\nu}$ build two Lorentz invariants quadratic in the fields $\mathbf{E}$ and $\mathbf{B}$.

b) Given a tensor $T^{\mu\nu}$ build a Lorentz invariant linear in this tensor.

3. *Stress-energy tensor*
The electromagnetic stress-energy tensor is given by

$$T^{\mu\nu} = \varepsilon_0 \left(F^{\mu}{}_{\alpha} F^{\nu\alpha} - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) , \tag{8}$$

where $F_{\mu\nu}$ is the field-strength tensor and $\eta_{\mu\nu}$ is the Minkowski metric.

a) Compute the trace $\eta_{\mu\nu} T^{\mu\nu}$.

b) Verify that

$$T^{00} = u = \frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \tag{9}$$

is the electromagnetic energy density.

c) Verify that

$$T^{0i} = \frac{1}{c} (\mathbf{S})_i \, , \tag{10}$$

where $\mathbf{S}$ is the Poynting vector. Notice that $\frac{1}{c} T^{0i}$ is also the momentum density (component i) in the electromagnetic field.

d) Using Maxwell equations (6) and (7) show that

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} j_\alpha F^{\alpha\nu} \, . \tag{11}$$

Rewrite the $\nu = 0$ component of this equation using the relations (9) and (10). What is the physical meaning of this equation? What about the spatial components $\nu = 1, 2, 3$?