

1. Reflection, Refraction and Fresnel Equations

Consider an electromagnetic wave incident on a plane interface separating two materials (linear dielectric and magnetic) at $z = 0$. Without loss of generality, choose your axes so that the incident ray is in the $x - z$ plane. The electric fields of the incident, reflected and the transmitted rays can be written in general as:

$$\begin{aligned}\vec{E}_I(\vec{r}, t) &= \vec{\mathcal{E}}_I e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)} \\ \vec{E}_R(\vec{r}, t) &= \vec{\mathcal{E}}_R e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)} \\ \vec{E}_T(\vec{r}, t) &= \vec{\mathcal{E}}_T e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}\end{aligned}\quad (1)$$

$\nabla \times \vec{E} = 0$ implies that $E_x^{in} = E_x^{out}$ and $E_y^{in} = E_y^{out}$ on all points on the interface.

- a) Begin by imposing the continuity condition at the point $x = y = z = 0$ to show that the frequencies are all the same. Now use the dispersion relation to deduce the relation between $|\vec{k}_I|$, $|\vec{k}_R|$ and $|\vec{k}_T|$.
- b) Now consider the continuity condition at a general point (x, y) on the interface to deduce the familiar laws of reflection and refraction:
 - i. $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ all lie in the same plane.
 - ii. The angle of incidence θ_I is equal to the angle of reflection θ_R .
 - iii. Snell's law $n_1 \sin \theta_I = n_2 \sin \theta_T$

(Hint: $Ae^{ia\xi} + Be^{ib\xi} = Ce^{ic\xi} \quad \forall \quad \xi \Rightarrow a = b = c$).

We now consider the polarization of the wave and we have two possibilities (see figure ??):

- Parallel polarization where $\vec{E}_I$ is in the plane of incidence ($x - z$ plane in our case).
- Perpendicular polarization where $\vec{E}_I$ is perpendicular to the plane of incidence (along y axis in our case).

Any incident wave can be written as a linear combination of the two.

- c) For the case of the parallel polarization, use the continuity of E_x and H_y to deduce the following relations between the incident, reflected and transmitted rays:

$$\frac{|\vec{\mathcal{E}}_R|}{|\vec{\mathcal{E}}_I|} = \frac{Z_2 \cos \theta_T - Z_1 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I} \quad ; \quad \frac{|\vec{\mathcal{E}}_T|}{|\vec{\mathcal{E}}_I|} = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I} \quad (2)$$

where $Z = \sqrt{\frac{\mu}{\epsilon}}$ is called the wave impedance.

- d) Specialize to the case of non-magnetic materials ($\mu = \mu_0$) and show that at the Brewster's angle, $\theta_B = \tan^{-1} \left(\frac{n_2}{n_1} \right)$, there is no reflection for parallel polarized light.
- e) Now consider the case of perpendicular polarization to get:

$$\frac{|\vec{\mathcal{E}}_R|}{|\vec{\mathcal{E}}_I|} = \frac{Z_2 \cos \theta_I - Z_1 \cos \theta_T}{Z_2 \cos \theta_I + Z_1 \cos \theta_T} \quad ; \quad \frac{|\vec{\mathcal{E}}_T|}{|\vec{\mathcal{E}}_I|} = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_I + Z_1 \cos \theta_T} \quad (3)$$

- f) Is it possible to have no reflection at some special angle in the case of perpendicular polarization for non-magnetic materials?

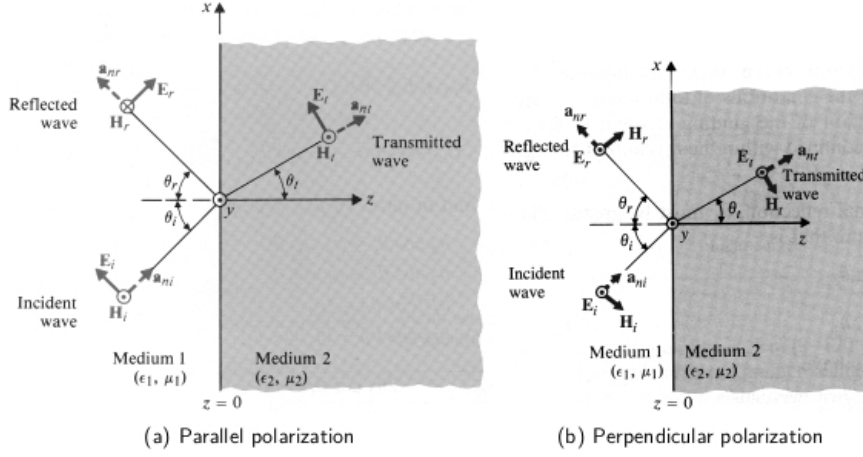


Figure 1: Courtesy D. K. Cheng, "Field and Wave Electromagnetics", 2nd. Ed., Addison Wesley

Solution

- a) Consider the x direction, the electric field at the point $x = y = z = 0$ is given by

$$\begin{aligned} E_x^{in} &= \mathcal{E}_{Ix}e^{-i\omega_I t} + \mathcal{E}_{Rx}e^{-i\omega_R t} \\ E_x^{out} &= \mathcal{E}_{Tx}e^{-i\omega_T t} \end{aligned} \quad (4)$$

The electric field along x direction must be continuous at all times and therefore we have

$$\mathcal{E}_{Ix}e^{-i\omega_I t} + \mathcal{E}_{Rx}e^{-i\omega_R t} = \mathcal{E}_{Tx}e^{-i\omega_T t} \quad \forall \quad t \quad (5)$$

This is only possible if $\omega_I = \omega_R = \omega_T$, hence the frequencies of the reflected and the transmitted wave is the same as that of the incident wave. The dispersion relation between the wave number and the frequency is given by

$$|\vec{k}| = \frac{n}{c}\omega \quad (6)$$

where n is the refractive index and c is the speed of light in vacuum. Using which, we arrive at the following relation:

$$|\vec{k}_I| = |\vec{k}_R| = \frac{n_1}{n_2}|\vec{k}_T| \quad (7)$$

- b) Once again, considering the electric field along the x direction, but this time at a general point on the $z = 0$ plane (x, y) we have

$$\mathcal{E}_{Ix} \exp[i(k_{Ix}x + k_{Iy}y)] + \mathcal{E}_{Rx} \exp[i(k_{Rx}x + k_{Ry}y)] = \mathcal{E}_{Tx} \exp[i(k_{Tx}x + k_{Ty}y)]$$

But this condition must be satisfied for all points (x, y) on the $z = 0$ plane.

i. This is only possible if $k_{Ix} = k_{Rx} = k_{Tx}$ and $k_{Iy} = k_{Ry} = k_{Ty}$. Thus $\vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ all in the same plane.
In our case, the incident wave is in the $x - z$ plane and therefore $k_{Ix} = |\vec{k}_I| \sin \theta_I$ and $k_{Iy} = 0$.

ii. For the reflected wave, we have $k_{Rx} = |\vec{k}_R| \sin \theta_R$ and $k_{Ry} = 0$. Using $|\vec{k}_I| = |\vec{k}_R|$, we immediately get $\theta_I = \theta_R$ ¹.

iii. For the refracted wave, $k_{Tx} = |\vec{k}_T| \sin \theta_T$ and $k_{Ty} = 0$. Using $|\vec{k}_I| = \frac{n_1}{n_2} |\vec{k}_T|$, we arrive at Snell's Law: $n_1 \sin \theta_I = n_2 \sin \theta_T$

c) For the case of parallel polarization, we have

$$\begin{aligned}\vec{\mathcal{E}}_I &= |\vec{\mathcal{E}}_I| (\cos \theta_I \hat{x} - \sin \theta_I \hat{z}) \\ \vec{\mathcal{E}}_R &= |\vec{\mathcal{E}}_R| (\cos \theta_I \hat{x} + \sin \theta_I \hat{z}) \\ \vec{\mathcal{E}}_T &= |\vec{\mathcal{E}}_T| (\cos \theta_T \hat{x} - \sin \theta_T \hat{z})\end{aligned}\tag{8}$$

We write the magnetic intensity $\vec{H}$ as $\vec{H}(\vec{r}, t) = \vec{\mathcal{H}} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ and compute its magnitude $|\vec{\mathcal{H}}| = \frac{1}{\mu} |\vec{\mathcal{B}}| = \frac{1}{Z} |\vec{\mathcal{E}}|$ where $Z = \sqrt{\frac{\mu}{\epsilon}}$

$$\begin{aligned}\vec{\mathcal{H}}_I &= \frac{1}{Z_1} |\vec{\mathcal{E}}_I| \hat{y} \\ \vec{\mathcal{H}}_R &= -\frac{1}{Z_1} |\vec{\mathcal{E}}_R| \hat{y} \\ \vec{\mathcal{H}}_T &= \frac{1}{Z_2} |\vec{\mathcal{E}}_T| \hat{y}\end{aligned}\tag{9}$$

Therefore from the continuity of E_x and H_y we get the following two equations:

$$\begin{aligned}|\vec{\mathcal{E}}_I| \cos \theta_I + |\vec{\mathcal{E}}_R| \cos \theta_I &= |\vec{\mathcal{E}}_T| \cos \theta_T \\ \frac{1}{Z_1} |\vec{\mathcal{E}}_I| - \frac{1}{Z_1} |\vec{\mathcal{E}}_R| &= \frac{1}{Z_2} |\vec{\mathcal{E}}_T|\end{aligned}\tag{10}$$

We can solve these two equations to get

$$\frac{|\vec{\mathcal{E}}_R|}{|\vec{\mathcal{E}}_I|} = \frac{Z_2 \cos \theta_T - Z_1 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I} \quad ; \quad \frac{|\vec{\mathcal{E}}_T|}{|\vec{\mathcal{E}}_I|} = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_T + Z_1 \cos \theta_I}\tag{11}$$

d) We now specialise to non-magnetic materials, $\mu = \mu_0$ ². Then $Z = \frac{\mu_0 c}{n}$ and notice that $|\vec{\mathcal{E}}_R| = 0$ if $Z_2 \cos \theta_T = Z_1 \cos \theta_I$, which becomes the condition

$$n_1 \cos \theta_T = n_2 \cos \theta_I\tag{12}$$

Recall that we also have Snell's Law

$$n_2 \sin \theta_T = n_1 \sin \theta_I\tag{13}$$

¹This is the unique solution since both θ_I and θ_R lie between 0 and $\frac{\pi}{2}$.

²This is a good approximation for most transparent materials, for example, water has $\frac{\mu}{\mu_0} = 0.999992$.

We solve for θ_I by squaring both equations and then using $\cos^2 \theta_T = 1 - \sin^2 \theta_T$ and then eliminating $\sin \theta_T$.

However, here is a quick way to get the answer, first multiply the two equations to get

$$\sin \theta_T \cos \theta_T = \sin \theta_I \cos \theta_I \quad (14)$$

which implies that $\sin(2\theta_T) = \sin(2\theta_I)$. For θ_I, θ_T between 0 and $\frac{\pi}{2}$, there are two possibilities:

$$\theta_T = \theta_I \quad \text{or} \quad \theta_T = \frac{\pi}{2} - \theta_I \quad (15)$$

The first possibility is ruled out since $n_2 \neq n_1$ ³ means that $\theta_T \neq \theta_I$ and therefore $\theta_T = \frac{\pi}{2} - \theta_I$. Using this in Snell's Law we arrive at Brewster's angle

$$\tan \theta_I = \frac{n_2}{n_1} \quad (16)$$

- e) For perpendicular polarization, we write the electric field and magnetic intensity:

$$\begin{aligned} \vec{\mathcal{E}}_I &= |\vec{\mathcal{E}}_I| \hat{y} \\ \vec{\mathcal{E}}_R &= |\vec{\mathcal{E}}_R| \hat{y} \\ \vec{\mathcal{E}}_T &= |\vec{\mathcal{E}}_T| \hat{y} \end{aligned} \quad (17)$$

and

$$\begin{aligned} \vec{\mathcal{H}}_I &= \frac{1}{Z_1} |\vec{\mathcal{E}}_I| (-\cos \theta_I \hat{x} + \sin \theta_I \hat{z}) \\ \vec{\mathcal{H}}_R &= \frac{1}{Z_1} |\vec{\mathcal{E}}_R| (\cos \theta_I \hat{x} + \sin \theta_I \hat{z}) \\ \vec{\mathcal{H}}_T &= \frac{1}{Z_2} |\vec{\mathcal{E}}_T| (-\cos \theta_T \hat{x} + \sin \theta_T \hat{z}) \end{aligned} \quad (18)$$

and once again impose continuity of E_x and H_y to get

$$\begin{aligned} |\vec{\mathcal{E}}_I| + |\vec{\mathcal{E}}_R| &= |\vec{\mathcal{E}}_T| \\ -\frac{1}{Z_1} |\vec{\mathcal{E}}_I| \cos \theta_I + \frac{1}{Z_1} |\vec{\mathcal{E}}_R| \cos \theta_I &= -\frac{1}{Z_2} |\vec{\mathcal{E}}_T| \cos \theta_T \end{aligned} \quad (19)$$

These two equations can be solved to give

$$\frac{|\vec{\mathcal{E}}_R|}{|\vec{\mathcal{E}}_I|} = \frac{Z_2 \cos \theta_I - Z_1 \cos \theta_T}{Z_2 \cos \theta_I + Z_1 \cos \theta_T} \quad ; \quad \frac{|\vec{\mathcal{E}}_T|}{|\vec{\mathcal{E}}_I|} = \frac{2Z_2 \cos \theta_I}{Z_2 \cos \theta_I + Z_1 \cos \theta_T} \quad (20)$$

- f) We see that the condition for no reflection is $Z_2 \cos \theta_I = Z_1 \cos \theta_T$ which for non-magnetic materials becomes

$$n_1 \cos \theta_I = n_2 \cos \theta_T \quad (21)$$

Snell's Law tells us that $n_1 \sin \theta_I = n_2 \sin \theta_T$ which implies that

$$\tan \theta_I = \tan \theta_T \quad (22)$$

The only solution to this equation is $\theta_I = \theta_T$ which is only possible if $n_1 = n_2$. Thus there is no special angle at which there is no reflection in the case of perpendicular polarization for the non-trivial case of $n_1 \neq n_2$.

³ $n_1 = n_2$ is the trivial case where there is no reflection and the wave just passes through without any deflection.