Quantum mechanics II, Problems 12 : Group averaging and Conjugacy classes

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Problem 1 : Conjugacy classes and number of irreducible representations

For the groups C3v and Z_N compute :

1. Their conjugacy classes.
2. The number of (non-equivalent) irreducible representations.
3. The possible dimensions of these irreducible representations.

Problem 2 : Group representation theory applied to dephasing

You already did the first two questions in the last exercise sessions but the answers are useful for the next questions.

1. Prove that the Pauli matrices and the identity (times ± 1, $\pm i$) form a (non-Abelian) group with the matrix product.
2. Prove that if $R(g)$ is a representation of a group G then $R(g) \otimes R(g)$ is also a representation of G.
3. Consider a unitary irreducible representation $R(g) = U_g$ of group G. Use the Grand Orthogonality Theorem to prove that

$$\frac{1}{N}\sum_g U_g X U_g^\dagger = \frac{1}{d}\,\mathrm{Tr}[X]\, I \tag{1}$$

   where $d = \dim(X)$ and N is the order of the group.
4. Use this result to (carefully!) explain why randomly applying either I (i.e, do nothing), σ_x, σ_y, or σ_z (with equal probability) to any single qubit state on average results in the maximally mixed state.
5. Consider now instead a completely reducible unitary representation $U_g = \oplus_k R_k(g)$ where the $R_k(g)$ are d_k dimensional unitary irreducible representations. It can be shown that

$$\langle X\rangle_G = \frac{1}{N}\sum_g U_g X U_g^\dagger = \frac{1}{d_k}\bigoplus_k \mathrm{Tr}[X\Pi_k]\Pi_k\,. \tag{2}$$

   What are Π_k and d_k in this expression?
6. The above relation for averaging over representations of finite groups, Eq. (2), generalizes to averaging over compact Lie groups. In this case the finite average $\frac{1}{N}\sum_g$ becomes a continuous integral over a uniform measure $\int d\mu(g)$ and we have :

$$\langle X\rangle_G := \int_G d\mu(g) U_g X U_g^\dagger = \frac{1}{d_k}\bigoplus_k \mathrm{Tr}[X\Pi_k]\Pi_k. \tag{3}$$

Use this result to derive an explicit expression (i.e. compute the relevant d_k and Π_k) for the averaged state ρ that results from randomly evolving ρ under the tensor product of two random single qubit unitaries. That is, from apply $U \otimes U$ with $U \in \mathrm{U}(2)$, to any two qubit state ρ, and then averaging :

$$\langle\rho\rangle = \int_{U(2)} d\mu\, U \otimes U\, \rho\, U^\dagger \otimes U^\dagger \ . \tag{4}$$

7. Hence (or otherwise) compute the states that result from averaging (i.e, compute $\langle\rho\rangle$ in Eq. (4)) for the following states :

   i. $\rho = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

   ii. $\rho = |\Psi^-\rangle\langle\Psi^-|$ with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

   iii. $\rho = |00\rangle\langle 00|$

   iv. An arbitrary tensor product two qubit state $\rho\otimes\sigma$ (hint : use the Bloch vector representation).