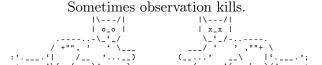
## Quantum mechanics II, Chapter 2: Entanglement (Part 3)

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## Problem 1 : Tsirelson's Bound

Suppose

$$Q = \mathbf{q} \cdot \sigma, \quad R = \mathbf{r} \cdot \sigma, \quad S = \mathbf{s} \cdot \sigma, \quad T = \mathbf{t} \cdot \sigma,$$

where  $\mathbf{q}, \mathbf{r}, \mathbf{s}$  and  $\mathbf{t}$  are real unit vectors in three dimensions. Let

$$A = Q \otimes S + R \otimes S + R \otimes T - Q \otimes T \tag{1}$$

Show that

$$A^2 = (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T].$$

Use this result to prove that

$$\langle A \rangle = \langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \le 2\sqrt{2}.$$

What is the maximum value of  $\langle A \rangle$  in the classical case? What do you infer from the difference between the classical and the quantum result?

Hint: Assume any classical observables Q, R, S, T with possible measurement outcomes  $\pm 1$ . In classical physics all observables commute with each other.

## Problem 2: Mermin-Peres and Telepathy (non-examinable)

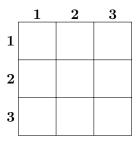


Figure 1 – Mermin-Peres Game Square

This exercise is another demonstration of quantum vs. classical correlations. We consider Alice and Bob playing a cooperative game on a  $3\times3$  square (see Figure 1). Alice will

be assigned a secret row, and Bob assigned a secret column. The goal of the game is for Alice and Bob to fill their column/row with +1 or -1, without seeing each other's values, while respecting the following constraints:

- Alice's row must have an even number of -1 (None or 2) i.e. the product of the entries is +1.
- Bob's column must contain an odd number of -1 (1 or 3) i.e. the product of the entries is -1.
- The square which is in both, Alice's row and Bob's column, must be filled with the same number by both (both +1 or both -1).

Alice and Bob are allowed to strategize before the game, however any communication is forbidden after they have been assigned their respective row or column.

- 1. Suppose they only have access to classical resources. Find a strategy that works out with a probability of success of 8/9 (It can be shown that this is the optimal success probability using classical resources). Hint: Try to fill as many squares as possible of the 3×3 square with +1 and -1, while respecting the given constraints. How many squares can you fill until you reach a contradiction?
- 2. What changes if Bob and Alice can communicate *after* learning the row/column they've been assigned?

	1	2	3
1	+1/ - 1	+1	+1
2	+1		
3	+1		

FIGURE 2 – Mermin-Peres Game Square after measuring  $Z \otimes Z \otimes Z \otimes Z$  and obtaining  $\{1,1,1,1\}$  as measurement outcome.

3. Now suppose Alice and Bob share two copies of a maximally entangled Bell state i.e. the full quantum state is  $|\psi\rangle = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle\right)$ . Each of them has one part of the entangled state, and they can choose to perform on each spin a measurement of  $\{X,Y,Z,\mathbb{1}\}$  (by "measuring  $\mathbb{1}$ " on spin i, it is meant that no measurement is performed on the i-th spin). The goal is to find a measurement strategy that beats the classical one in terms of success chances. Before the game starts, they start to think about a strategy to win the game and come up with the following idea:

They want to agree upon measurements they can perform on their qubits, for every combination of row and column they could possibly get. If they find 9 measurements (for every combination of row and column) that will successfully fill their row and columns (successful means they agree in the ij-th square, while keeping their constraints), they would win the game always. They start trying out different measurements they could perform. For example: If Alice got row 1 and Bob got column 1 they would have to agree in the top left corner of the  $3\times3$  square. They decide to try both measure Z on both their qubits i.e. they measure  $Z\otimes Z\otimes Z\otimes Z$  on  $|\psi\rangle$ . This results in 4 different possibilities to fill the squares: If Alice and Bob both measure  $\{1,1\}$ , the state collapsed to  $|0000\rangle$ . Alice will put  $\{1,1\}$  in the second and third square of the first row, and Bob will put  $\{1,1\}$  in the second and third square of the first column. To respect the constraints Alice needs to fill another 1 in the top left square while Bob has to fill a -1 (see Figure 2). With this measurement they would therefore lose the game. Had they measured

 $\{1, 1, -1, -1\}$  (corresponding to the collapse to  $|0011\rangle$ ), Alice would fill again  $\{1, 1\}$  while Bob would fill  $\{-1, -1\}$ . Again respecting the constraints, Alice has to fill a 1, while Bob needs to fill a -1. They again lose.

Can you find a 4-qubit measurement that will result in a win every time, without failure?

4. Do the same for every combination of rows and columns Alice and Bob can obtain.

## Problem 3: Bloch sphere for pure or mixed states of a two-level system

In the lecture you have started to talk about density matrices. This exercise serves as a first introduction to the topic, connecting to the already known concept of the Bloch sphere.

1. Derivation of Bloch vector from generic pure state. Any pure one-qubit quantum state can be written as ket

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle \quad \theta \in [0, \pi), \ \phi \in [0, 2\pi)$$

or as the density matrix,

$$|\psi\rangle\langle\psi| = \frac{1}{2} \left(\mathbb{1} + \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{r}\right)$$

Find an expression for  $\mathbf{r}$  in terms of  $\theta$  and  $\phi$ . What does the vector  $\mathbf{r}$  denote?

- 2. Derivation of Bloch vector from properties of density operators. Define the set of density matrices with the following 3 conditions:
  - The density matrix is Hermitian :  $\hat{\rho}^{\dagger} = \hat{\rho}$
  - It has trace  $1 : \operatorname{Tr} \hat{\rho} = 1$
  - It is positive or null:  $\langle \Psi | \hat{\rho} | \Psi \rangle \geq 0$ ,  $\forall \Psi$

Show that any density matrix  $\hat{\rho}$  of the 2 level system can be written

$$\hat{\rho} = \frac{1}{2} (\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}), \tag{2}$$

where  $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ . Argue that  $\boldsymbol{r}$  is a real vector of 3D space and  $|\boldsymbol{r}| \leq 1$ .

- 3. Show that the state is pure iff ||r|| = 1. Explain why  $\text{Tr}[\rho^2]$  is a measure of the 'purity' of a quantum state.
- 4. Sketch on the Bloch sphere the states:

  - b)  $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ c)  $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle |1\rangle)$

  - d)  $\frac{1}{\sqrt{3}}(|0\rangle i\sqrt{2}|1\rangle)$ .

  - e)  $\frac{1}{2}$ 11 f)  $\frac{1}{3}$  $|+\rangle\langle+|+\frac{2}{3}|-\rangle\langle-|$
- 5. Give a geometric argument to show that  $\frac{1}{2}(|+\rangle\langle+|+|-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle0|+|1\rangle\langle1|)$ . (Is this surprising?)

**Disclaimer:** think about the meaning of this state. How would you represent  $|0\rangle\langle 0|$  on the Bloch sphere? And  $|1\rangle\langle 1|$ ? Now if the space of qubits is a convex space, what is the point in the Bloch sphere that represents the combination of the previous one?

Now make the same reasoning for  $|+\rangle \langle +|, |-\rangle \langle -|$ .