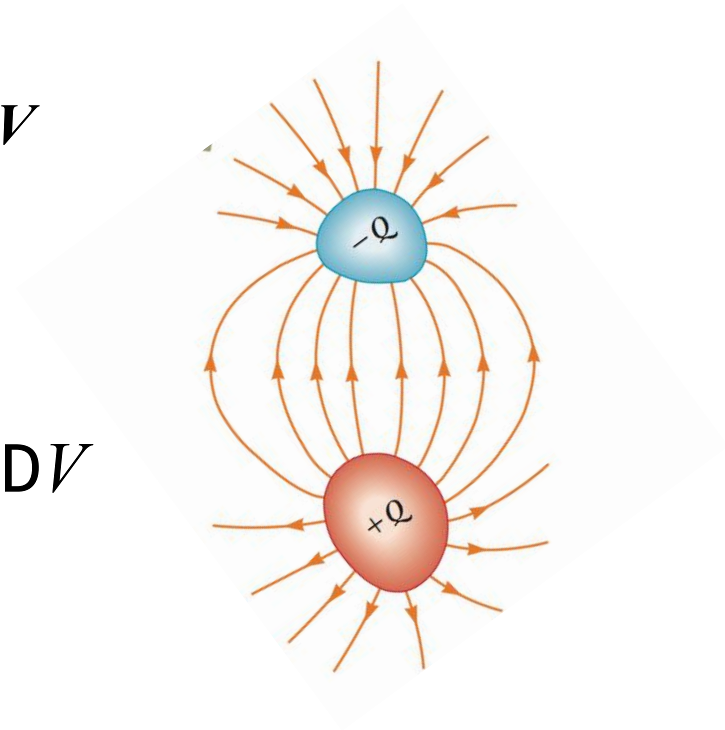
Week 5:
Electric Capacitance

Definition of Capacitance

Consider two closely-spaced conductive objects, charged to +Q and –Q. Each has its own potential $\pm V$



<u>Experiments</u> show that $Q \propto \mathrm{D}V$

The proportionality constant, ***C***, is called the capacitance of this system of two charges: $Q = C \cdot \mathrm{D}V$

Such a system is called a *CAPACITOR* $C = \dfrac{Q}{\mathrm{D}V}$

- *Capacitance is a measure of how much charge can be stored in a capacitor at a given ΔV*
- *Capacitance (capacity) is a **positive** scalar value*

The unit of capacitance is ***farad*** (F): 1 **F**= 1 C/V

- *In practical electronics smaller derivatives of F are in use:*

1 pF = 10^{-12} F
1 nF = 10^{-9} F
1 µF = 10^{-6} F

Capacitance of a two-conductor capacitor

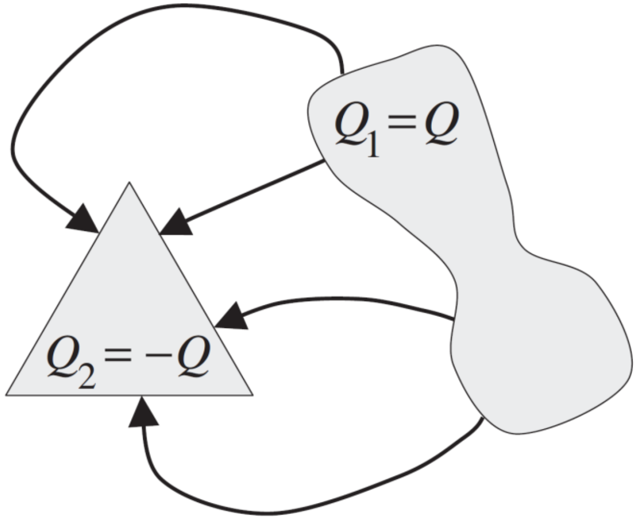
(mutual capacitance)

The capacitance of a two−conductor capacitor is also a **purely geometric quantity** defined as :

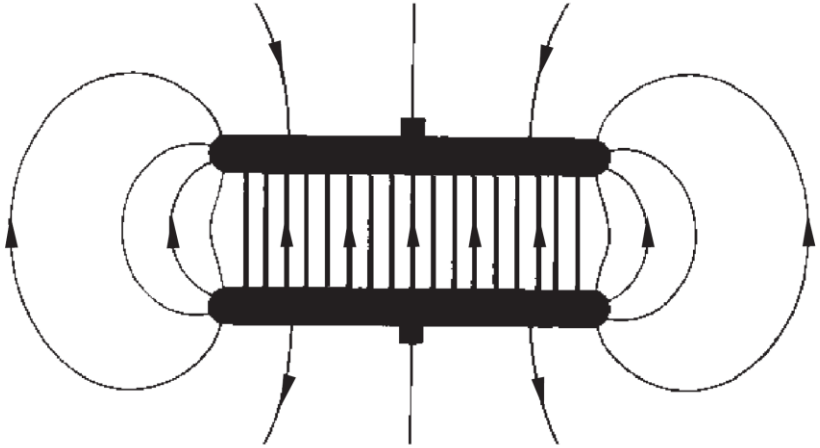
$$\boxed{C \equiv \frac{Q}{V}}$$

Q: total charge on one of the two conductors

V: potential difference between the two conductors

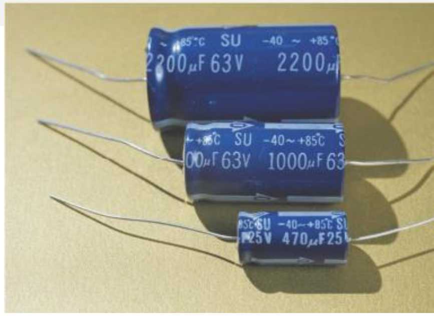


Generic two conductor capacitor



Two conductor parallel plate capacitor

Capacitor and Capacitance



Capacitor: device that stores electric potential energy and electric charge.

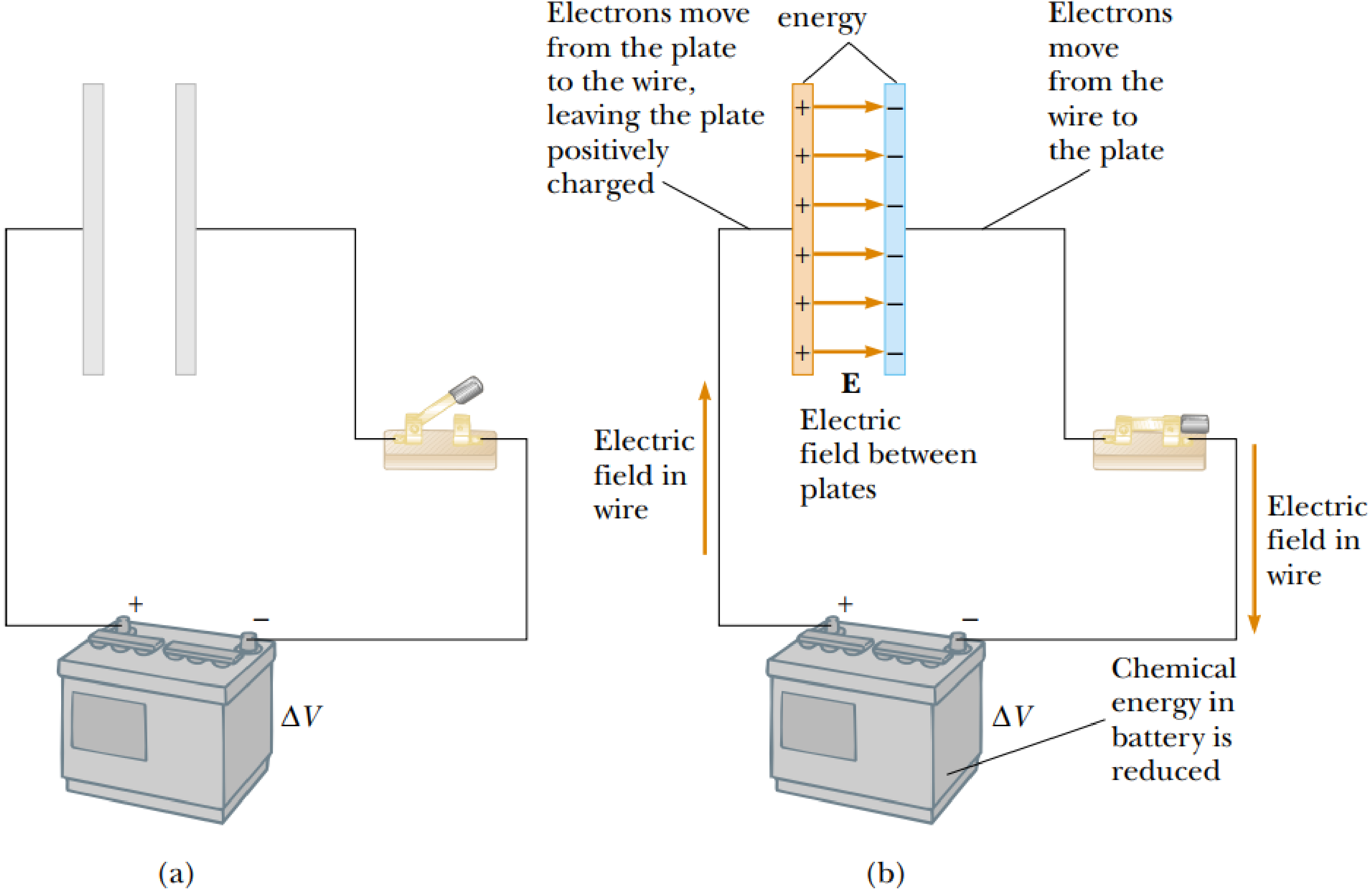
- Two conductors separated by an insulator form a capacitor.
- The net charge on a capacitor is zero.

Figure shows a battery connected to a single parallel-plate capacitor with a switch in the circuit.

When the switch is closed, the battery establishes an electric field in the wires and charges flow between the wires and the capacitor.

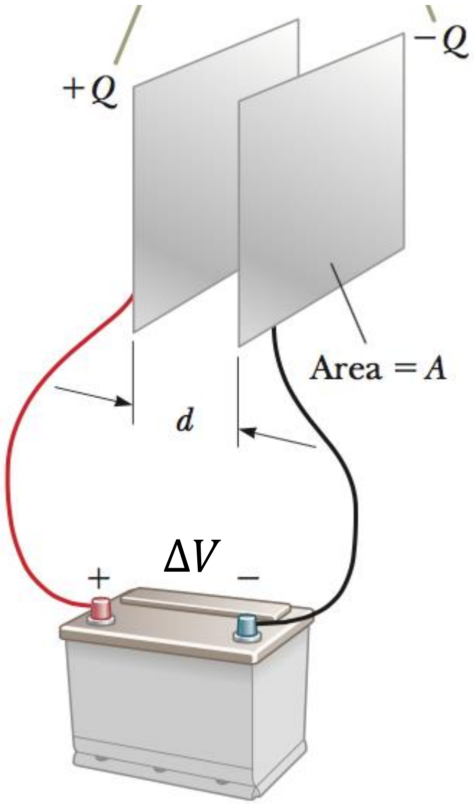
As this occurs, there is a transformation of energy within the system. Before the switch is closed, energy is stored as chemical energy in the battery. When the switch is closed, some of the chemical energy in the battery is converted to electric potential energy related to the separation of positive and negative charges on the plates.

As a result, we can describe a capacitor as a device that stores energy as well as charge.



Parallel plate capacitor

- The electric potential of each plate is as of the battery terminals
- The battery moves electrons to keep **V** fixed:
- Charges the plate of the capacitor.
- uniform electric field between the plates,
- uniformly distributed charge over opposite surfaces



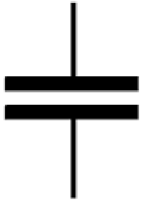
- *Conventionally, the charge on **one** of the plates is the charge of a capacitor **Q***

For each of the two plates:

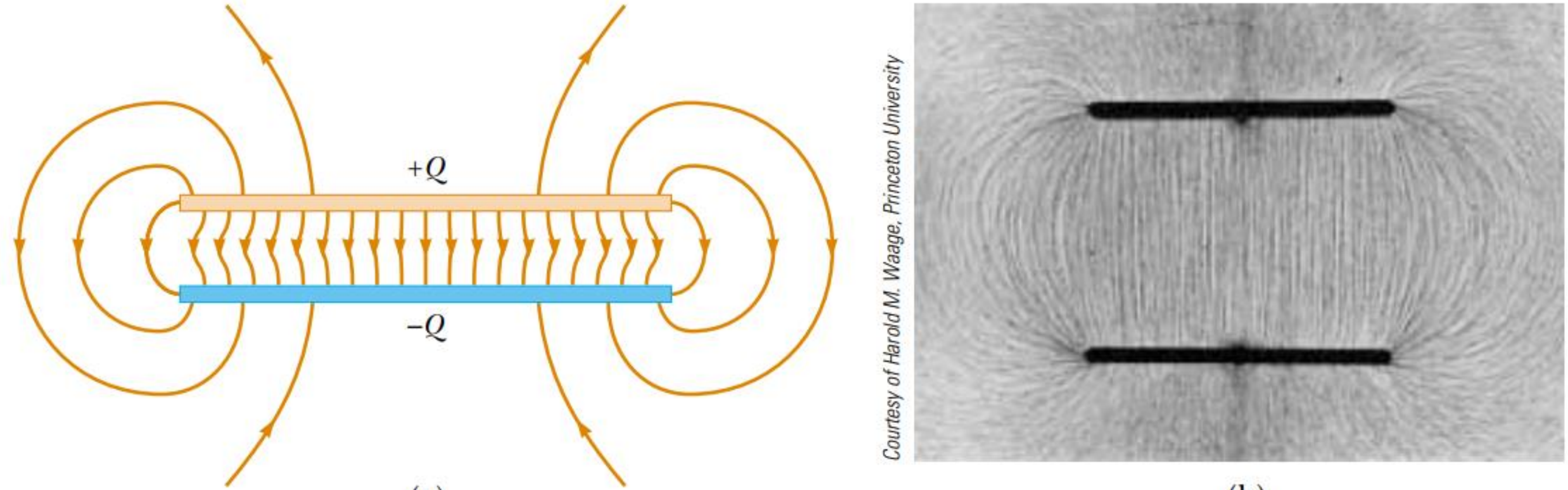
(Gauss) $E_1 = \frac{s}{2 \cdot e_0}$ ⇨ $E_0 = \frac{\sigma}{\varepsilon_0}$ (between the plates)

⇨ Potential difference: $\Delta V \equiv V = E_0 d = \frac{Qd}{\varepsilon_0 A}$

Capacitance: $\boxed{C = \frac{Q}{DV} = \frac{Q \cdot e_0 A}{Qd} = \frac{e_0 A}{d}}$



Edges effect in real capacitors



(a) (b)

Courtesy of Harold M. Waage, Princeton University

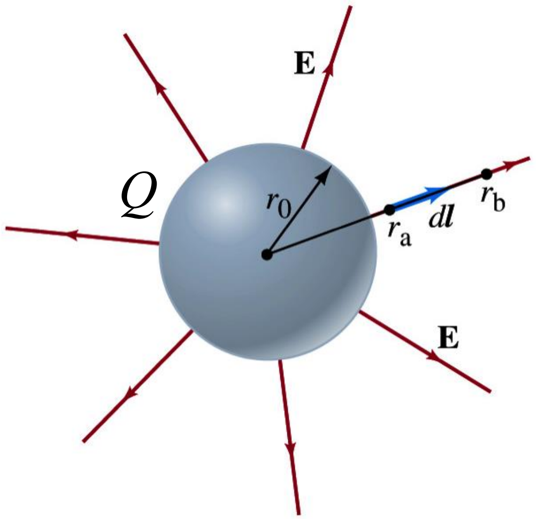
(a) The electric field between the plates of a parallel-plate capacitor is uniform near the center but nonuniform near the edges. (b) Electric field pattern of two oppositely charged conducting parallel plates. Small pieces of thread on an oil surface align with the electric field.

A careful inspection of the electric field lines for a parallel-plate capacitor reveals that the field is uniform in the central region between the plates, as shown in Figure. However, the field is nonuniform at the edges of the plates. Such end effects can be neglected if the plate separation is small compared with the length of the plates.

Capacitance of a single conductor capacitor
(self-capacitance)

Occasionally you will hear someone speaking about the capacitance of a single conductor.
In this case the second conductor (with the opposite charge) is an imaginary sperical shell of infinite radius surrounding the conductor.

Exemple 1. Capacitance of a capacitor with one conductor: conductive sphere



$$V=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r_0}=\frac{1}{4\pi\varepsilon_0}\frac{CV}{r_0}\quad\Rightarrow\quad C=4\pi\varepsilon_0 r_0$$

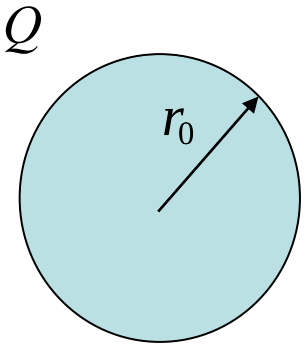
$r_0 = 10\ \mu\text{m} \Rightarrow C \cong 10^{-15}$ F=1 fF

$r_0 = 1\ \text{mm} \Rightarrow C \cong 10^{-13}$ F=0.1 pF

$r_0 = 1\ \text{m} \Rightarrow C \cong 10^{-10}$ F

$r_0 = 6.3\times10^{6}\ \text{m} \Rightarrow C \cong 10^{-3}$ F (capacitance of the Earth)

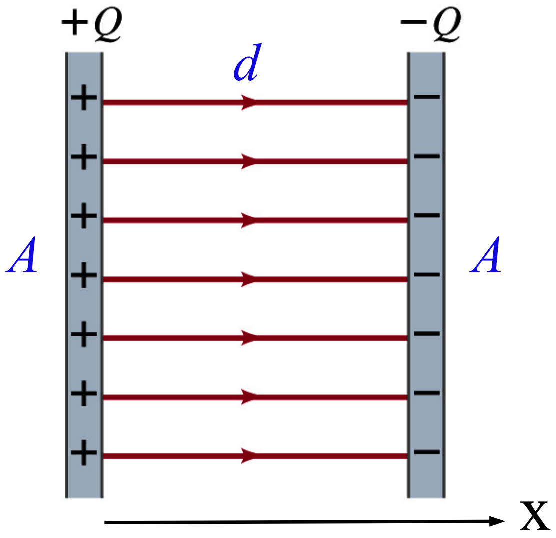
Exemple 2. Capacitance of a capacitor with one conductor: conductive disk with radius r_0



The potential V is constant on the disk but the charge density is not uniform.
Without demonstration:

$$V=\frac{Q}{8\varepsilon_0 r_0}=\frac{CV}{8\varepsilon_0 r_0}\quad\Rightarrow\quad C=8\varepsilon_0 r_0$$

Exemple 3. Capacitance of a capacitor with two parallel plate conductors



$$E=\frac{\sigma}{\varepsilon_0}=\frac{Q}{A\varepsilon_0}\ ; \quad V(x)-V(0)=-\int_0^x E dx=-\frac{Q}{A\varepsilon_0}x\ ; \quad 0\le x\le d$$

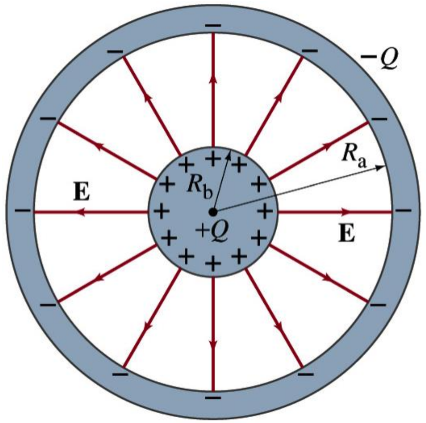
$$\Rightarrow$$

$$\Delta V=\frac{Q}{A\varepsilon_0}d \qquad \text{mais } C\equiv\frac{Q}{V}$$

$$\Rightarrow$$

$$C=\varepsilon_0\frac{A}{d}$$

Exemple 4. Capacitance of a “spherical” two-conductor capacitor



$$E(r)=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \qquad R_b<r<R_a \qquad \Rightarrow$$

$$V(r)-V(R_a)=-\int_{R_a}^{r}\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}dr=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{R_a}\right) \qquad R_b<r<R_a$$

$$\Delta V=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_b}-\frac{1}{R_a}\right) \qquad \text{mais } C\equiv\frac{Q}{V}$$

$$\Rightarrow$$

$$C=4\pi\varepsilon_0\frac{R_aR_b}{R_a-R_b}$$

Capacitor in series

The total potential is the sum of potentials across each capacitor

- Same charge (Q).

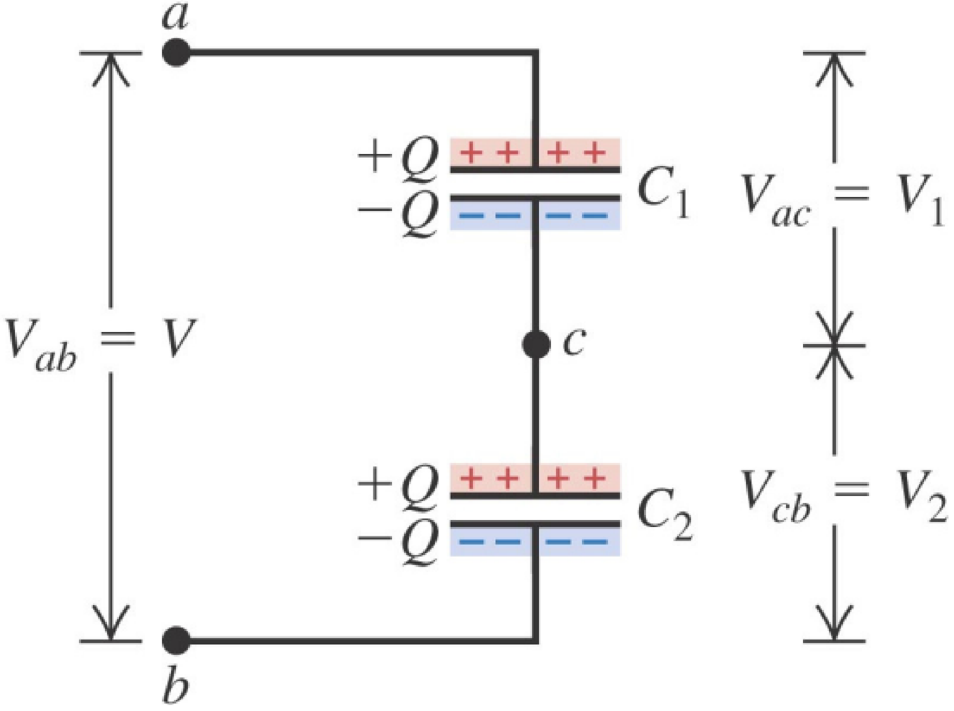
$$\boxed{V_{ab} = V_{ac} + V_{cb}}$$

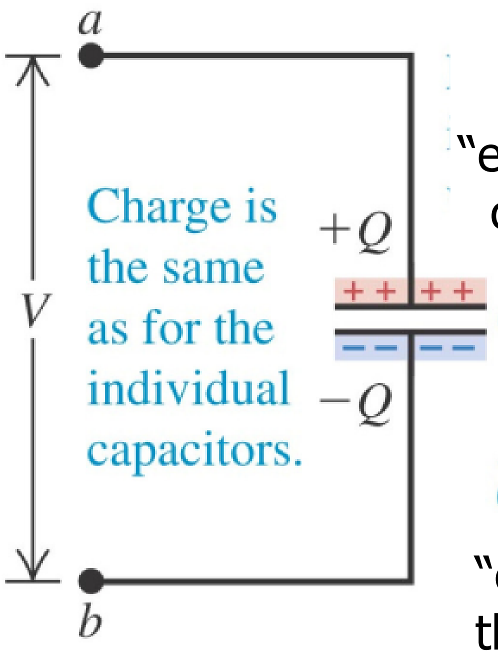
The same charge on each capacitor

Total charge is also Q

$$C_{eq} = \frac{Q}{V_{ab}} = \frac{Q}{V_1 + V_2} \Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{V_1}{Q} + \frac{V_2}{Q} = \frac{1}{C_1} + \frac{1}{C_2}} \Rightarrow \boxed{\frac{1}{C_{tot}} = \sum_{i=1}^{N} \frac{1}{C_i}}$$

Generalization to *N* capacitors in series





Charge is the same as for the individual capacitors.

"equivalent" capacitor

$$C_{eq} = \frac{Q}{V}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

"equivalent" means "stores the same total charge if the voltage is the same."

Capacitor in parallel

- Same potential V, different charge.

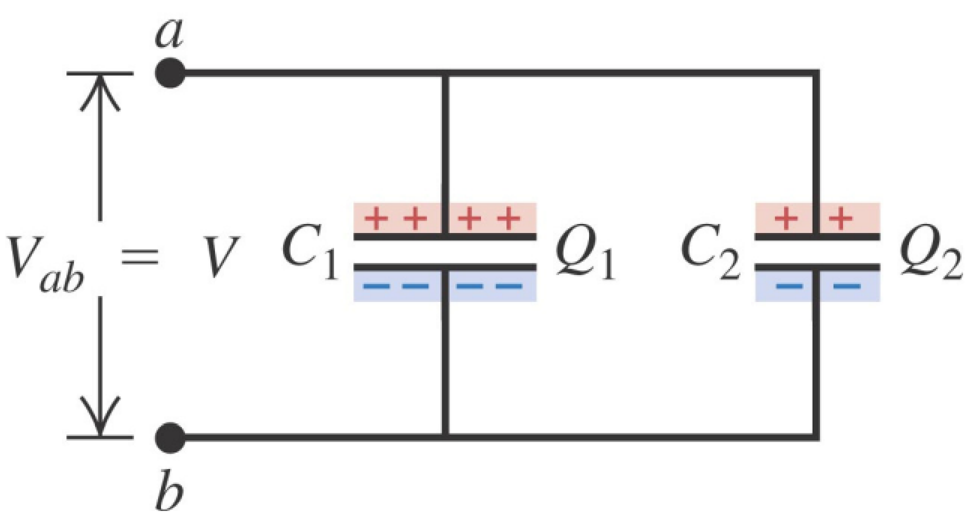
$\boxed{Q_1 = C_1 V_1}$ $\boxed{Q_2 = C_2 V_2}$

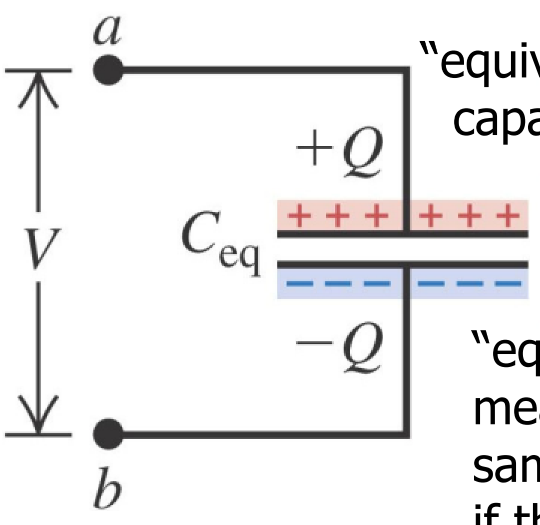
$\boxed{Q = Q_1+Q_2}$

Same potential difference across all 2 capacitors
The total charge is the sum of the individual charges

$$C_{eq} = \frac{Q}{V_{ab}} = \frac{Q_1+Q_2}{V} \Longrightarrow \boxed{C_{eq} = \frac{Q_1}{V} + \frac{Q_2}{V} = C_1 + C_2} \Longrightarrow \boxed{C_{tot}^{II} = \sum C_i}$$

Generalization to *N* capacitors in series





"equivalent" capacitor

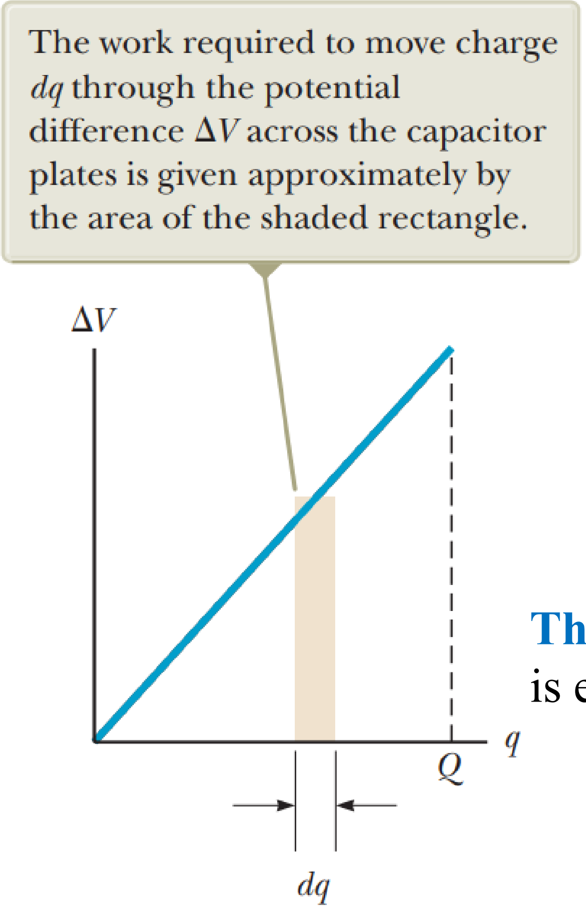
"equivalent" means "stores the same total charge if the voltage is the same."

Energy Stored in a Charged Capacitor

Consider a capacitor with charge +/-q
How much work is needed to bring a positive charge *dq* from the negative plate to the positive plate?
NB: we are charging the capacitor

$$\Delta V = q/C.$$

$$dW = \Delta V\, dq = \frac{q}{C}\, dq$$


The work required to move charge *dq* through the potential difference Δ*V* across the capacitor plates is given approximately by the area of the shaded rectangle.

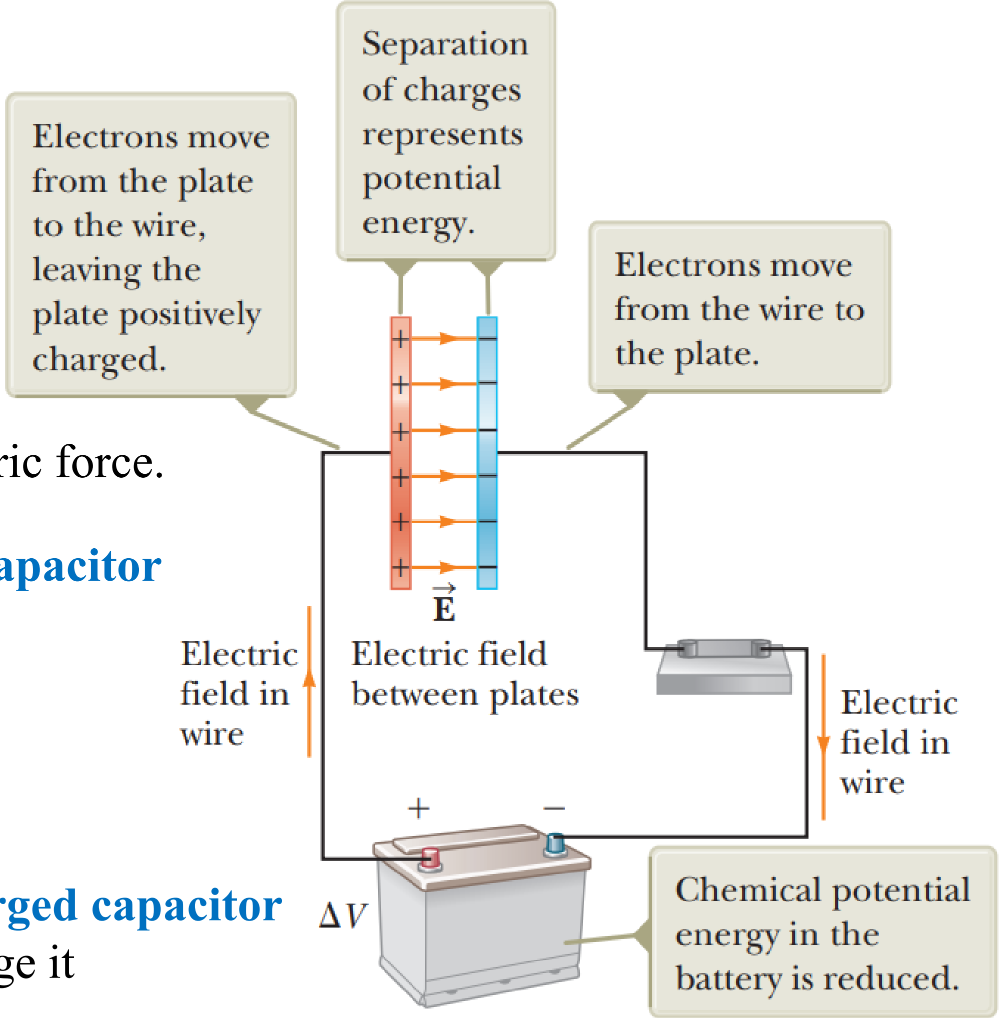
This work has to be done against the electric force.

The total work required to charge the capacitor from q=0 to some final charge q=Q is

$$W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

The electric potential energy stored in a charged capacitor is equal to the amount of work required to charge it

$$\boxed{U_E = \frac{Q^2}{2C} = \tfrac{1}{2}Q\,\Delta V = \tfrac{1}{2}C(\Delta V)^2}$$


Separation of charges represents potential energy.

Electrons move from the plate to the wire, leaving the plate positively charged.

Electrons move from the wire to the plate.

Chemical potential energy in the battery is reduced.

The battery moves electrons to keep V fixed: it charges the plates of the capacitor

EPFL

Energy Stored in a Charged Capacitor: Electric-Field Energy

- We can consider ***the energy in a capacitor to be stored in the electric field created between the plates as the capacitor is charged***. This description is reasonable because the electric field is proportional to the charge on the capacitor.
- For a parallel-plate capacitor, the potential difference is related to the electric field through the relationship $\Delta V = \boldsymbol{E}d$. Furthermore, its capacitance is $C = \varepsilon_0 A/d$.
- Substituting these expressions into $U_E = \frac{1}{2}C(\Delta V)^2$(see previous slide)

$$U_E = \tfrac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2 = \tfrac{1}{2}(\epsilon_0 Ad)E^2$$

Because the *volume* occupied by the electric field is Ad, the energy per unit volume $\boldsymbol{u_E = U_E/Ad}$, known as *energy density*, is:

$$\boxed{u_E = \frac{1}{2}\epsilon_0 E^2}$$

Electric Energy Density (vacuum)

Although this Equation was derived for a parallel-plate capacitor, the expression is generally valid regardless of the source of the electric field. That is, ***the energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point***.

Energy stored in capacitor vs. energy stored in battery

12 V, 100 Ah car battery

- charge: $3.6\text{x}10^5$ C, energy: $4.3\text{x}10^6$ J

100 μF capacitor at 12 V

- charge: $Q=CV= 1.2\text{x}10^{-3}$ C, energy: $U=CV^2/2=7.2\text{x}10^{-3}$ J

If batteries store so much more energy, why use capacitors?

- capacitor stores charge physically, battery stores charge chemically
- capacitor can **release** stored charge and energy **much faster**

DEMO https://auditoires-physique.epfl.ch/experiment/422

Energy stored in a capacitor

final state

$$U_{pot} = W_{ext} = \int_0^Q V\, dQ = \int_0^V CV\, dV = \frac{CV^2}{2}$$

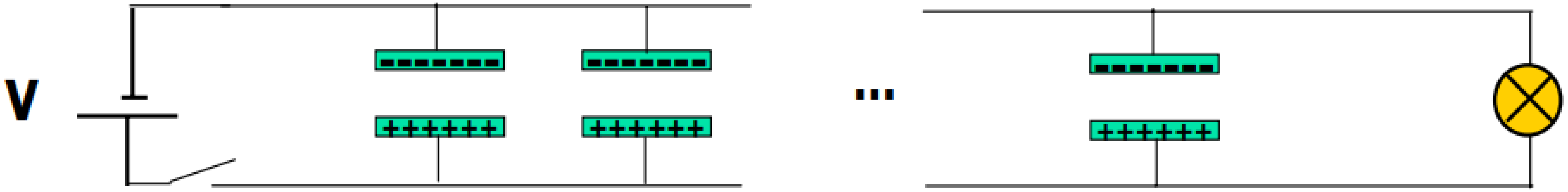
initial state

$$Q = C \cdot V$$

$$dQ = C \cdot dV$$

DEMO capa

10^6 joules of energy are stored at high voltage in capacitor banks and released during a period of a few milliseconds. The enormous current produces incredibly high magnetic fields.



Capacitors with Dielectrics

The potential difference is measured by a device called a *voltmeter*. **If a dielectric is now inserted between the plates** as in Figure (b), the voltmeter indicates that the **voltage** between the plates **decreases** to a value ΔV

$\textit{if } Q_o = Q \textit{ is constant (no battery connected)}$:

$$Q_o = \Delta V_o C_o = Q = C\Delta V \qquad \Delta V = \frac{\Delta V_0}{\kappa}$$

$$C/C_o = \Delta V_0/\Delta V = \kappa$$

The dimensionless factor ***κ (or K) is called the dielectric constant of the material***. The dielectric constant varies from one material to another. In this section, we analyze this change in capacitance in terms of electrical parameters such as electric charge, electric field, and potential difference;

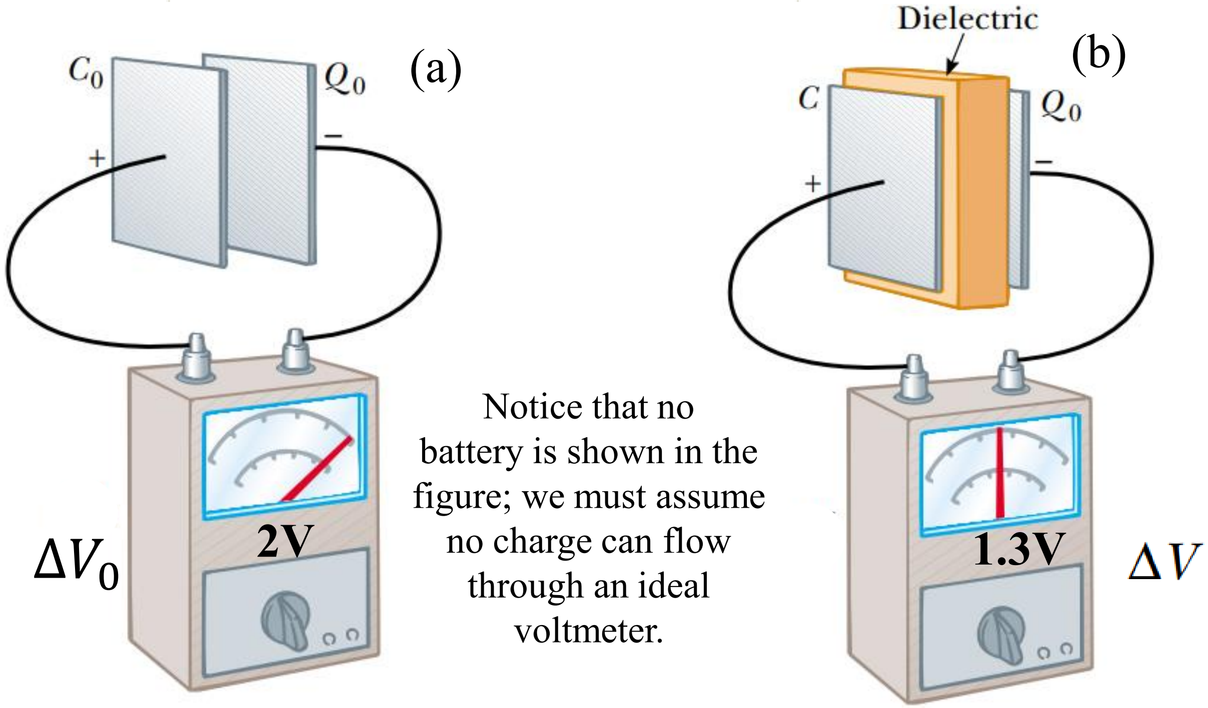
$$C = \frac{Q_0}{\Delta V} = \frac{Q_0}{\Delta V_0/\kappa} = \kappa\frac{Q_0}{\Delta V_0} \qquad C = \kappa C_0$$

That is, the capacitance increases by the factor κ when the dielectric completely fills the region between the plates

$$\text{Because } C_0 = \epsilon_0 A/d \Longrightarrow C = \kappa\frac{\epsilon_0 A}{d}$$

The potential difference across the charged capacitor is initially ΔV_0.

After the dielectric is inserted between the plates, the charge remains the same, but the potential difference decreases and the capacitance increases.



Notice that no battery is shown in the figure; we must assume no charge can flow through an ideal voltmeter.

Hence, there is no path by which charge can flow and alter the charge on the capacitor.

Induced Charge and Polarization

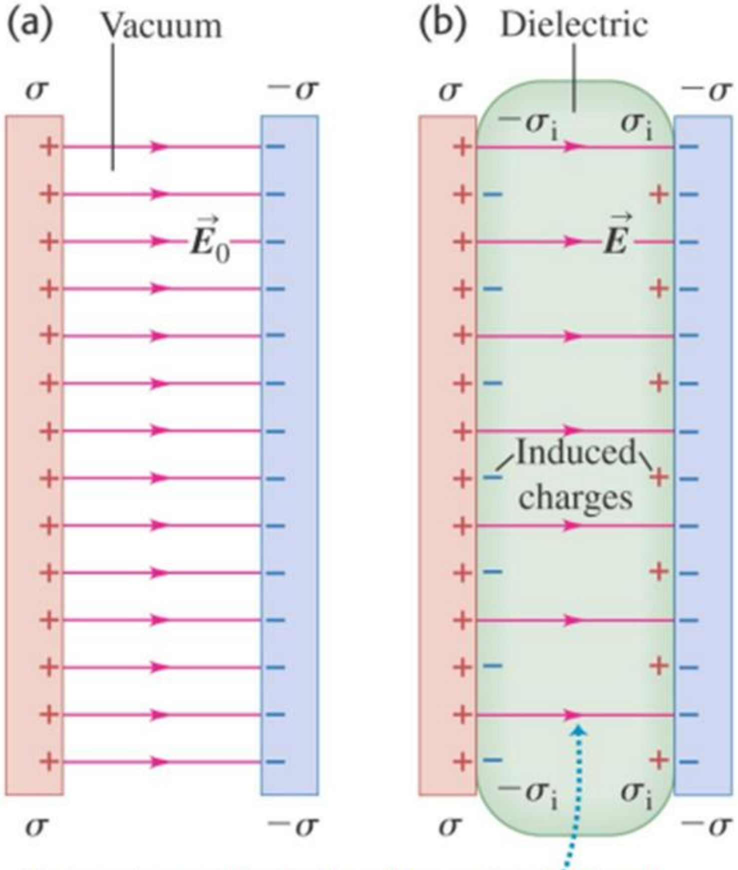
E = field with the dielectric between plates

E_0 = field with vacuum between the plates

- E is smaller when the dielectric is present because the total surface charge density smaller.
- The surface charge on conducting plates does not change, but an induced charge of opposite sign appears on each surface of the dielectric.
- The dielectric remains electrically neutral (only charge redistribution).

Polarization:

redistribution of charge within a dielectric.

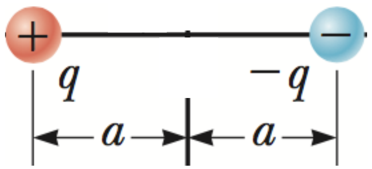


For a given charge density σ, the induced charges on the dielectric's surfaces reduce the electric field between the plates.

Field lines change in the presence of dielectrics.

Capacitors with Dielectrics

- Opposite to conductors, has NO free electrons
- Model: collection of dipoles (fixed but can rotate):

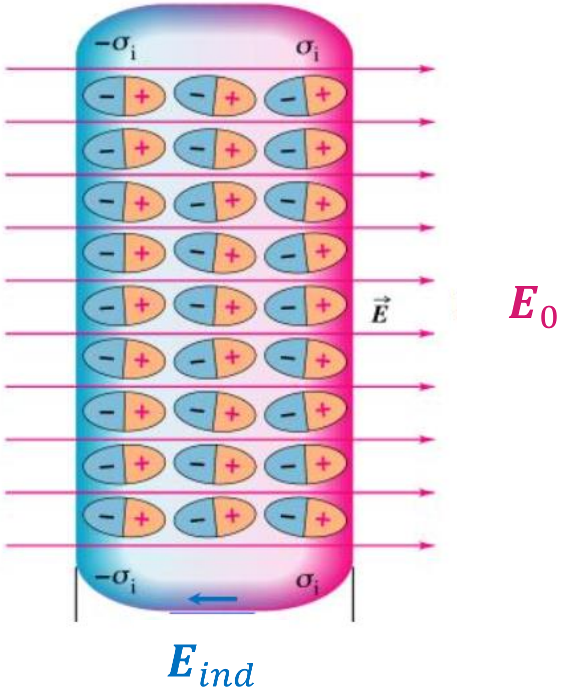


- Normally randomly oriented (E=0)
  - External electric field forces them to be oriented:

Polarization

- Surface becomes charged with $\pm\sigma_i$ =>
- give rise to an induced electric field $\boldsymbol{E}_{ind}$ directed *against* the external field $\boldsymbol{E}_0$
- the field in the presence of a dielectric is

$$\boldsymbol{E} = \boldsymbol{E}_0 - \boldsymbol{E}_{ind}$$



***The electric field in dielectric* E *(i.e. the total electric field in the parallel plate capacitor, if the dielectric is filling the entire space between the plates)* becomes weaker than external field**

Capacitors with Dielectrics

Electric field is weaker in dielectric:

$$E = \frac{1}{K}E_0 = \frac{\varepsilon_0}{\varepsilon}E_0$$

E_0 is the electric field in the capacitor without dielectric

$$E_0 = \frac{Q}{\varepsilon_0 A}$$

K = dielectric constant

$\varepsilon = K\varepsilon_0$ permittivity of the dielectric

ε_0 = Vacuum permittivity

The induced surface charges on the dielectric give rise to an induced electric field E_{ind}, so: $E = E_0 - E_{ind}$

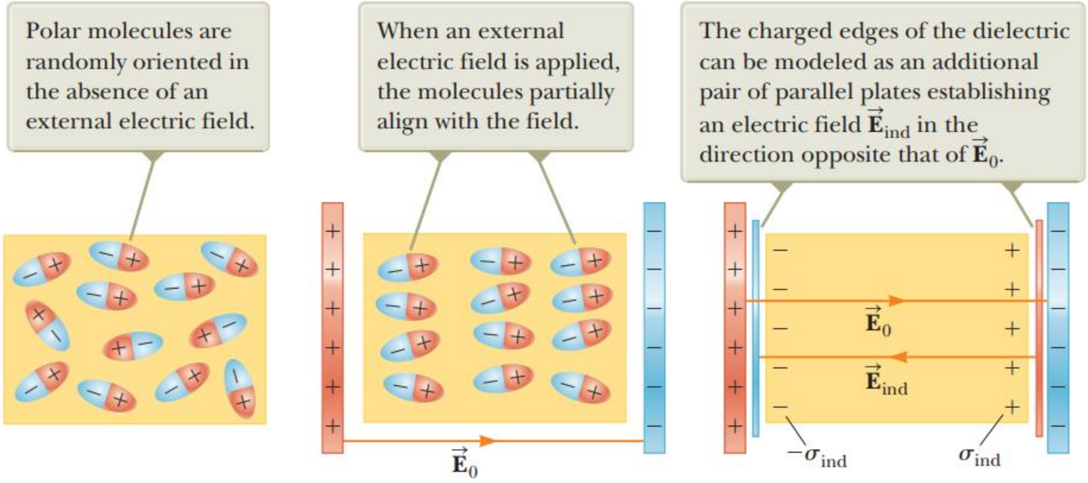
$$E = \frac{\varepsilon_0}{\varepsilon}E_0 = \frac{\varepsilon_0}{\varepsilon}\frac{Q}{\varepsilon_0 A} = \frac{Q}{\varepsilon A} = \frac{Q}{\varepsilon A} = \frac{Q}{K\varepsilon_0 A} = \frac{\sigma}{K\varepsilon_0}$$

$\Delta V = E\ d$ is voltage dropp across the capacitor with the dielectric

$$C = \frac{Q}{\Delta V} = \frac{Q}{E\ d} = \frac{Q}{d}\frac{\varepsilon A}{Q} = \frac{Q}{d}\frac{K\varepsilon_0 A}{Q} \Rightarrow \boxed{C = \frac{\varepsilon A}{d} = \frac{K\varepsilon_0 A}{d}}$$

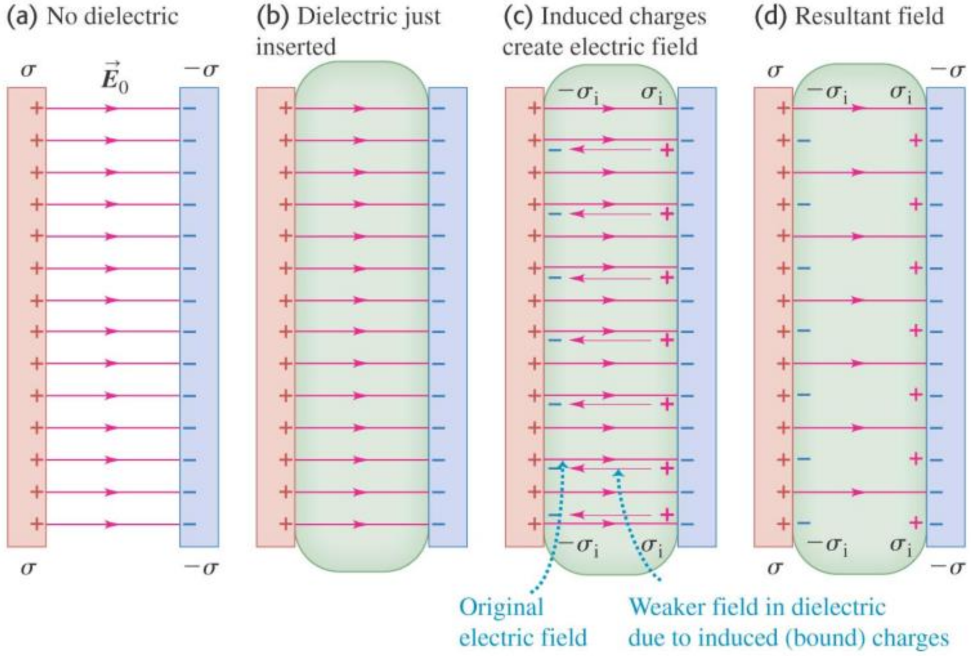
$$\frac{\sigma}{K\varepsilon_0} = \frac{\sigma}{\varepsilon_0} - \frac{\sigma_{ind}}{\varepsilon_0} \Rightarrow \boxed{\sigma_{ind} = \frac{(K-1)}{K}\sigma}$$

Polarization of a dielectric in electric field gives rise to bound charges on the surfaces, creating $+\sigma_{ind}$ and $-\sigma_{ind}$.


Polar molecules are randomly oriented in the absence of an external electric field.

When an external electric field is applied, the molecules partially align with the field.

The charged edges of the dielectric can be modeled as an additional pair of parallel plates establishing an electric field $\vec{E}_{ind}$ in the direction opposite that of $\vec{E}_0$.


(a) No dielectric

(b) Dielectric just inserted

(c) Induced charges create electric field

(d) Resultant field

Original electric field

Weaker field in dielectric due to induced (bound) charges

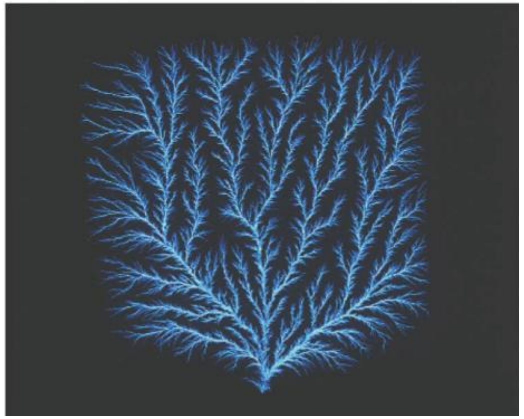
Table 24.1 Values of Dielectric Constant K at 20°C

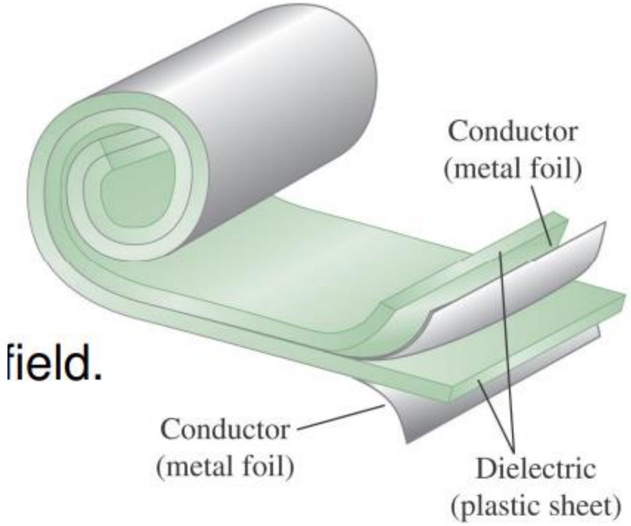
Material	K	Material	K
Vacuum	1	Polyvinyl chloride	3.18
Air (1 atm)	1.00059	Plexiglas	3.40
Air (100 atm)	1.0548	Glass	5–10
Teflon	2.1	Neoprene	6.70
Polyethylene	2.25	Germanium	16
Benzene	2.28	Glycerin	42.5
Mica	3–6	Water	80.4
Mylar	3.1	Strontium titanate	310

The dielectric layer increases the maximum potential difference between the plates of a capacitor and allows to store more Q.

Dielectric breakdown: partial ionization of an insulating material subjected to a large electric field.

A very strong electrical field can exceed the strength of the dielectric to contain it.





Summary

K is the dielectric constant of the material

Net charge on capacitor plates: $(\sigma-\sigma_i)$ (with σ_i = induced surface charge density)

$$E_0 = \frac{\sigma}{\varepsilon_0} \qquad E = \frac{E_0}{K} = \frac{\sigma - \sigma_i}{\varepsilon_0}$$

E = field with the dielectric between plates
E 0 = field with vacuum between the plates

Induced surface charge density: $$\boxed{\sigma_i = \sigma\left(1 - \frac{1}{K}\right)}$$

Permittivity of the dielectric: $$\boxed{\varepsilon = K\varepsilon_0}$$

Electric field (dielectric present): $$E = \frac{\sigma}{\varepsilon}$$

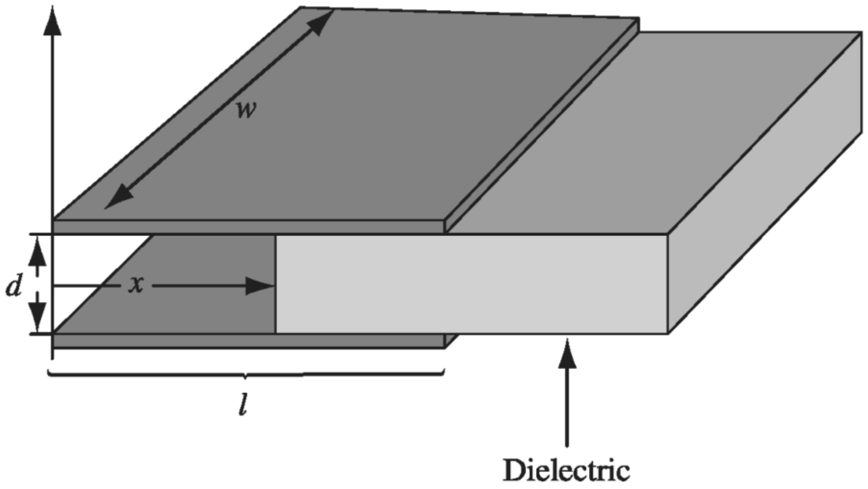
C = capacitance with the dielectric inside the plates of the capacitor
C 0 = capacitance with vacuum between the plates

Capacitance of parallel plate capacitor (dielectric present): $$\boxed{C = K \cdot C_0 = K\varepsilon_0 \frac{A}{d} = \varepsilon \frac{A}{d}}$$

Electric energy density (dielectric present): $$\boxed{u = \frac{1}{2} K\varepsilon_0 E^2 = \frac{1}{2}\varepsilon \cdot E^2}$$

5. Electrostatic energy (and force) in a capacitor partially filled with a dielectric.

Dielectric materials are attracted to electric fields due to the orientation of the **electric dipoles** (polarization of the molecules).
(**Conductive** materials are attracted to electric fields due to **charge redistribution**)

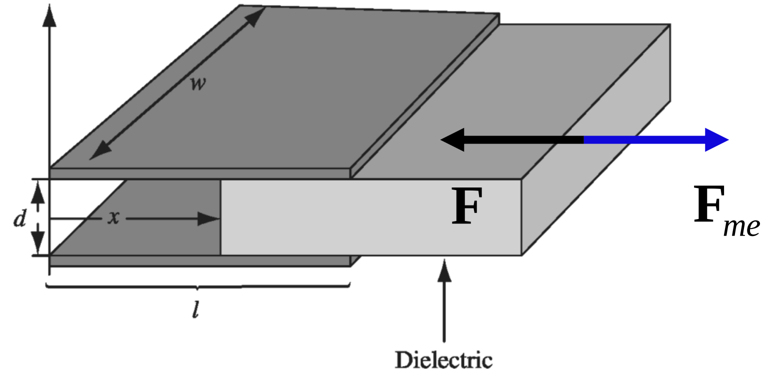


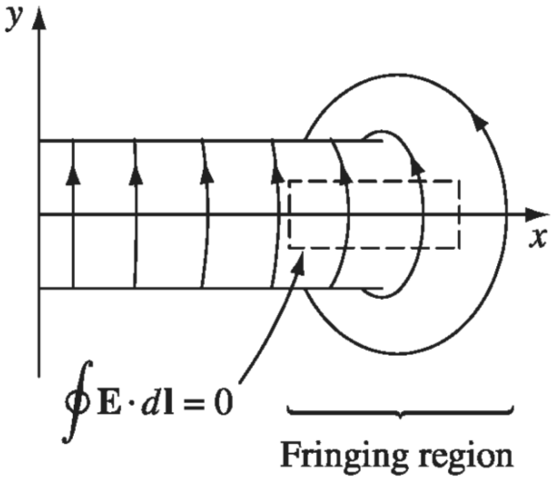
$$C(x) = \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon_0 \varepsilon_r w(l-x)}{d} = \frac{\varepsilon_0 w}{d}\left(x\left(1-\varepsilon_r\right)+\varepsilon_r l\right) = \frac{\varepsilon_0 w}{d}\left(\varepsilon_r l - \chi_e x\right)$$

$$\varepsilon_r = 1 + \chi_e \Rightarrow 1 - \varepsilon_r = -\chi_e$$

$$U_E(x) = \frac{1}{2} C(x) V(x)^2 = \frac{1}{2}\frac{Q(x)^2}{C(x)}$$

$$Q = CV \Rightarrow V = Q / C$$





If the electric field were perfectly homogeneous, there would be no force on the dielectric plate.The force on the dielectric plate is due to the electric fieldnon-homogeneous located in the ‘fringe region’.

$\mathbf{F}_{me}$: Force I have to apply to the dielectric plate
$\mathbf{F} = -\mathbf{F}_{me}$: Force on the Dielectric Plate

<u>Q constant:</u>

$dU_E = F_{me}dx$ $F_{me}dx$: Work I have to do to move the dielectric from dx,

$$\Rightarrow F = -\frac{dU_E}{dx} \qquad \Rightarrow$$

$$F = -\frac{d}{dx}\frac{1}{2}\frac{Q^2}{C(x)} = \frac{1}{2}\frac{Q^2}{C(x)^2}\frac{dC(x)}{dx} = \frac{1}{2}V^2\frac{dC(x)}{dx} = -\frac{\varepsilon_0\chi_e w}{2d}V^2$$

<u>V constant (a battery is always connected to the capacitor):</u>

$dU_E = F_{me}dx + VdQ$ $F_{me}dx$: Work I have to do to move the dielectric from dx,
VdQ: Work done by voltage source.

$$F = -\frac{dU_E}{dx} + V\frac{dQ}{dx} = -\frac{1}{2}V^2\frac{dC(x)}{dx} + V^2\frac{dC(x)}{dx} = \frac{1}{2}V^2\frac{dC(x)}{dx} = -\frac{\varepsilon_0\chi_e w}{2d}V^2$$

Therefore:
1) The force exerted on the dielectric does not depend on whether "Q" or "V" is held constant.
2) The force on the dielectric plate is towards the inside of the capacitor (towards the negative "x" direction.

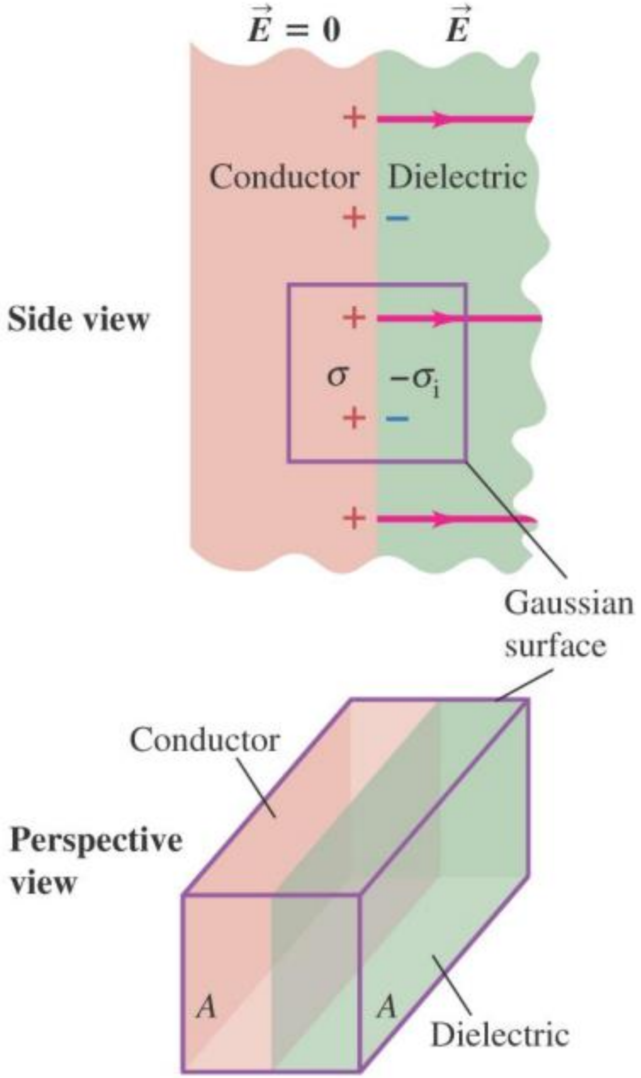
Gauss’s Law in Dielectrics

In the presence of dielectrics, Gauss’s Law must be modified to account for the bound charges.

$$EA = \frac{Q_{encl}}{\varepsilon_0} = \frac{(\sigma - \sigma_i)A}{\varepsilon_0}$$

$$\sigma_i = \sigma\left(1 - \frac{1}{K}\right)$$

$$EA = \frac{\sigma \cdot A}{K\varepsilon_0}$$



Flux through Gaussian surface (enclosed free charge / ε_0)

$$KEA = \frac{\sigma \cdot A}{\varepsilon_0}$$

Gauss Law in a dielectric:

$$\boxed{\oint K\vec{E} \cdot d\vec{A} = \frac{Q_{encl-free}}{\varepsilon_0}}$$