General Physics: Electromagnetism, Correction 4

Exercise 1 :

Calculate the Electric Field within and outside a charged cylinder of radius R and length L with a charge density obeying the law $\rho(r) = \rho_0(a - br)$, where a and b are arbitrary parameters. Consider the case where $L \gg R$, which means that the direction of the electric field lines are radial.

- **Hint 1**: remember that the electric field is radial, so try to take a proper Gaussian surface to easily compute the electric field flux (left part of Gauss's law);
- **Hint 2**: to compute the integral of the charge density, move to cylindrical coordinates

Solution 1 :

The Gauss law states that the flux of the electric field on a surface ∂V is given by the total charge Q contained within the volume V divided by ε_0. In formulas

$$\int_{\partial V} \vec{E} \cdot d\vec{\sigma} = \frac{1}{\varepsilon_0} \int_V d\vec{r}\, \rho(\vec{r}) = \frac{Q_{\text{tot}}}{\varepsilon_0}, \tag{1}$$

where $d\vec{\sigma}$ is the surface element, $d\vec{r}$ is the volume element and $\rho(\vec{r})$ is the charge density.

In the present case, the electric field is parallel to the surface element, and constant across the area, so that the left integral simply reduces to the area of the cylinder times the electric field, $2\pi r L E(r)$.

Now we have to compute the total charge within the cylinder. The integral is not so simple as the surface integral, because the charge density explicitly depends on the coordinates. We need to distinguish two cases:

1. $r < R$:

   The total charge within the gaussian surface is found the following way:

$$\begin{gathered} Q_{\text{tot}} = \int d\vec{r} \rho = \int_{-\frac{L}{2}}^{\frac{L}{2}} ds \int_0^{2\pi} d\phi \int_0^r dr' \rho(r') r' = \\ 2\pi\rho_0 L \int_0^r dr'(a - br')r' = 2\pi\rho_0 L \left(a\frac{r^2}{2} - b\frac{r^3}{3} \right) \end{gathered} \tag{2}$$

   Finally, we find the expression for the Electric Field :

$$2\pi r L E(r) = \frac{2\pi\rho_0 L}{\varepsilon_0} \left(a\frac{r^2}{2} - b\frac{r^3}{3} \right) \longrightarrow \boxed{E(r) = \frac{\rho_0}{\varepsilon_0} \left[\frac{ar}{2} - \frac{br^2}{3} \right]} \tag{3}$$

2. $r > R$:

In this case to find the total charge Q_{tot}, the charge density ρ needs to be integrated over the whole cylinder:

$$\oint \overrightarrow{E} \cdot dA = \frac{Q_{tot}}{\varepsilon_0} \to 2\pi r L E(r) = \frac{2\pi \rho_0 L}{\varepsilon_0} \int_0^R dr'(a - br')r' \tag{4}$$

And the expression for the Electric Field is:

$$2\pi r L E(r) = \frac{2\pi \rho_0 L}{\varepsilon_0} \left(a\frac{R^2}{2} - b\frac{R^3}{3} \right) \longrightarrow \boxed{E(r) = \frac{\rho_0}{\varepsilon_0} \left[\frac{aR^2}{2} - \frac{bR^3}{3} \right] \frac{1}{r}} \tag{5}$$

We notice that the Electric Field outside the cylinder is proportional to $1/r$. This is an expected result, as, sufficiently far away from the charge distribution, the electric field coincides with the electric field generated by a narrow wire, that is $\propto 1/r$.

Exercise 2 :

Consider a spherical cavity of radius a at the center of a non-conductive sphere of radius R. The volume charge density in the rest of the sphere varies according to $\rho = A/r$, where A is a positive constant. Determine the electric field for $a < r < R$.

- **Hint**: the charge inside a shell of thickness dr is $dq = \rho dV = \rho(4\pi r^2)dr$.

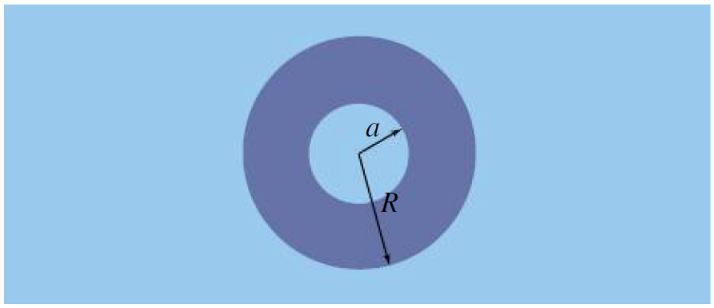

Figure 1: Spherical cavity of radius a at the center of a non-conductive sphere of radius R.

Solution 2 :

To solve this exercise we use again the Gauss law in Eq. (1). In this case, the Gaussian surface we have to consider is a sphere instead of a cylinder. As before, the electric field is radial and the left part of Eq. (1) gives the surface of the sphere times the electric field, $4\pi r^2 E(r)$. To compute the right part of Eq. (1), it is convenient to use spherical coordinates:

$$\int_V d\vec{r}\rho(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^2\theta \int_a^r dr' r'^2 A/r' = 2A\pi(r^2 - a^2). \tag{6}$$

By imposing the Gauss law, we can easily find the electric field for $a < r < R$:

$$E(r) = \frac{A}{2\varepsilon_0}\left(1 - \frac{a^2}{r^2}\right). \tag{7}$$

Notice that for $r = a$ the electric field is zero, because no charge is enclosed within the Gaussian surface.

Exercise 3 :

Two non-conducting spheres of radii R_1 and R_2 are uniformly charged with charge densities ρ_1 and ρ_2, respectively. They are separated at center-to-center distance a (see below). Find the electric field at point P located at a distance r from the center of sphere 1 and is in the direction θ from the line joining the two spheres assuming their charge densities are not affected by the presence of the other sphere.

- **Hint**: Work one sphere at a time and use the superposition principle.

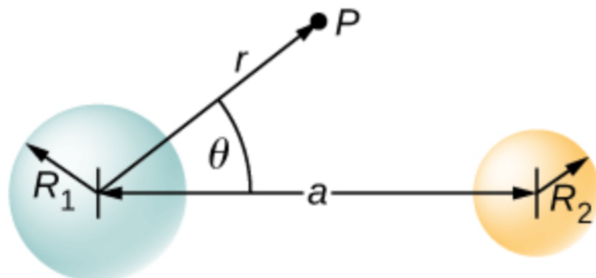

Figure 2: Two non-conducting spheres of radii R_1 and R_2, uniformly charged with charge densities ρ_1 and ρ_2, respectively.

Solution 3 :

To solve this exercise we proceed in the following way: we find the electric field generated by a single sphere and then we use the superposition principle to compute the total electric field (that is, the total electric field is give by the vectorial sum in the plane of the two electric fields).

To find the electric field outside the sphere R_ℓ, $\ell = 1, 2$, we apply the Gauss law, given by Eq. (1). Since the charge densities do not depend on the coordinates the total charge for the ℓ-th sphere is simply given by $Q_\ell = \rho_\ell V_\ell = \frac{4}{3}\pi R_\ell^3 \rho_\ell$ As a Gaussian surface, we use a sphere of radius r. We find

$$4\pi r^2 E_\ell(r) = \frac{4\pi}{3\varepsilon_0} R_\ell^3 \rho_\ell, \tag{8}$$

that is

$$E_\ell(r) = \frac{R_\ell \rho_\ell}{3\varepsilon_0} \frac{1}{r^2} = \frac{c_\ell}{r^2}. \tag{9}$$

To compute the total electric field in P, we must perform the vectorial sum of the two electric fields, $\vec{E} = \vec{E}_1 + \vec{E}_2$. First of all we know that the distance of P from the first sphere is given by r. We call the distance of P from the second sphere, r'. We call the angle between r' and a, ϕ.

We must compute r' in terms of θ and r. To do this we can use the Carnot theorem (or law of cosines), obtaining

$$r' = \sqrt{r^2 + a^2 - 2ar\cos\theta}. \tag{10}$$

We need also the angle ϕ as a function of r and θ. This can be determined by computing the height of the triangle connecting the two spheres and P. This height is given by $h = r\sin\theta$. Now the angle ϕ can be expressed in terms of h and the adjacent edge, that is $a - r\cos\theta$, as

$$\phi = \mathrm{atan}\left(\frac{r\sin\theta}{a - r\cos\theta}\right) \tag{11}$$

Now, the total electric field can be written as $\vec{E} = \vec{E}_1 + \vec{E}_2$, where

$$\begin{aligned} \vec{E}_1 &= E_1\cos\theta\,\hat{e}_x + E_1\sin\theta\,\hat{e}_y = \frac{c_1}{r^2}(\cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y), \qquad (12)\\ \vec{E}_2 &= -E_2\cos\phi\,\hat{e}_x + E_2\sin\phi\,\hat{e}_y = \frac{c_2}{r'^2}(-\cos\phi\,\hat{e}_x + \sin\phi\,\hat{e}_y). \end{aligned}$$

We finally obtain

$$\vec{E} = \hat{e}_x\left(\frac{c_1\cos\theta}{r^2} - \frac{c_2\cos\phi}{r'^2}\right) + \hat{e}_y\left(\frac{c_1\sin\theta}{r^2} + \frac{c_2\sin\phi}{r'^2}\right), \tag{13}$$

where $c_\ell = R_\ell\rho_\ell/3\varepsilon_0$ and r' and ϕ are given by Eqs. (10) and (11) respectively.

Exercise 4 :

A wire having a uniform linear charge density λ is bent into the shape shown in Fig.1. Find the electric potential and the electric field at point O.

- **Hint 1**: use the superposition principle and compute the total potential as the sum of three separate contributions in the three different regions;
- **Hint 2**: express the infinitesimal charge dq in the curved region as a function of the infinitesimal angle $d\theta$;
- **Hint 3**: for the electric field computation, use the symmetry of the wire to cancel out some contributions.

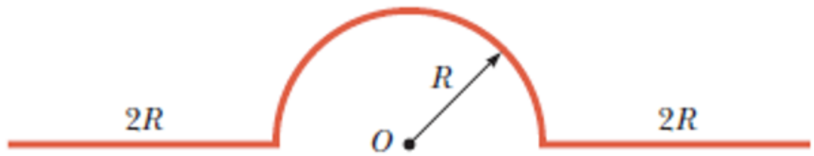

Figure 3: A bent wire with uniform charge density λ.

Solution 4 :

The charge in a small element of the straight part of the wire is given by $dq = \lambda dx$ and in the semi-circular part of the wire is given by $dq = \lambda R d\theta$. The potential in point O is found by decomposing the contributions of the two straight parts and the semi-circle.

$$V_O = \frac{1}{4\pi\varepsilon_0}\left(\int_R^{3R} \frac{\lambda dx}{|x|} + \int_0^{\pi} \frac{\lambda R d\theta}{R} + \int_R^{3R} \frac{\lambda dx}{|x|}\right) \tag{14}$$

$$V_O = \frac{1}{4\pi\varepsilon_0}\left(\lambda ln(x)\Big|_R^{3R} + \frac{\lambda \pi R}{R} + \lambda ln(x)\Big|_R^{3R}\right) \tag{15}$$

$$\boxed{V_O = \frac{1}{4\pi\varepsilon_0}\lambda(\pi + 2ln(3))} \tag{16}$$

To find the electric field E we **cannot** use $\overrightarrow{E} = -\nabla V(\overrightarrow{r})$ because $V(\overrightarrow{r})$ is unknown. Remember, we only found the expression of the potential at O. Therefore we need to calculate the electric field E directly at the origin O. By symmetry, the contributions of the two straight parts of the wire cancel each other. For every small element of the semi-circle, we can define $d\overrightarrow{E} = dE_x\hat{e}_x + dE_y\hat{e}_y$. By symmetry, we observe that the x component cancels. Hence, we only need to determine the y component:

$$dE_{y,O} = -\frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\sin(\theta) = -\frac{1}{4\pi\varepsilon_0}\frac{\lambda R d\theta}{R^2}\sin(\theta) \tag{17}$$

And then we integrate between 0 and π:

$$E_{y,O} = -\int_0^{\pi} \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{R^2}\sin(\theta)d\theta = \boxed{-\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{R}} \tag{18}$$

Exercise 5 :

Time of flight mass spectrometer (TOF MS) determines mass-to-charge ratio of ions by measuring time of their flight in a field free region (see Fig.3). It consists of a metal plate **A**, two metal grids **B** and **C**, which are transparent for ions, and a Detector. The grids **B** and **C** are grounded (i.e potential V=0), the plate **A** can be put at the fixed positive potential $V_0 = +1000\ V$ by closing the switch S. The ion Detector, which is very close to **C**, is at some high negative potential. Positively singly charged ions are initially very close to the plate **A** (they do not interact with each other). At a well-defined time $t = 0$ the switch S is **on** and V_0 is being applied to **A** and ions begin to move toward the detector.

(a) Show that ions of different masses ($q = +1$) will arrive to the detector at different time. Derive an expression for time-of-flight.

(b) Estimate the time-resolution of the detector, required to distinguish masses of uranium isotopes: ^{238}U ($m_1 = 3.95 \cdot 10^{-25}$ kg) and ^{235}U ($m_2 = 3.90 \cdot 10^{-25}$ kg).

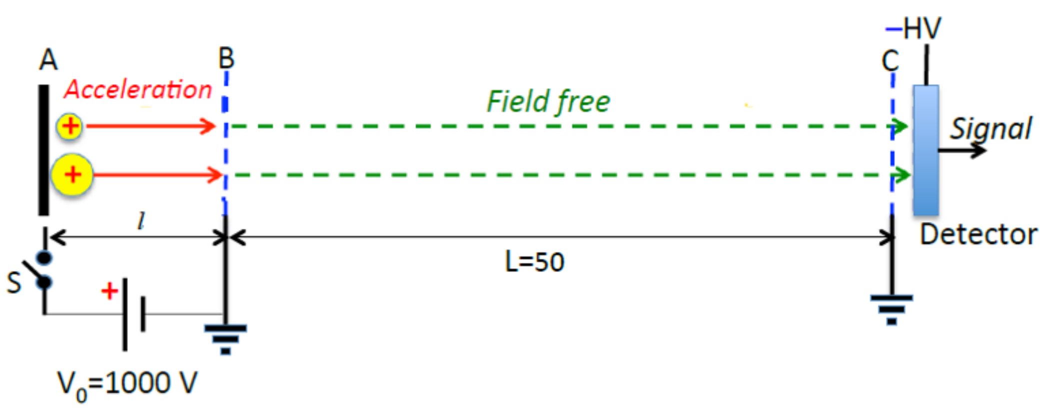

Figure 4: Time of flight-based mass spectrometer schematic.

Solution 5 :

The ions of different masses are accelerated between A and B and obtain the same kinetic energy qV_0 at B, which means that they will have different velocities. They then move each with its fixed attained velocity through the much longer free-field region **B–C**. Because the attained velocities are different the travel time through the field-free region **B–C** will differ for the ions of different masses. After C, the ions are immediately attracted by the detector. The ions with different masses therefore will appear at detector and be detected at different times (relative to the time t0 of switching S to On).

(a) In the acceleration region **A–B** a positively charged ion attains kinetic energy equal to the potential of the ion near the plate A:

$$E_{kin} = E_{el} \longrightarrow \frac{1}{2}mv^2 = qV_0 \tag{19}$$

Thus the ions obtain the velocity v at B:

$$v = \sqrt{\frac{2qV_0}{m}} \tag{20}$$

This velocity remains constant in the region **B-C** because there is no electric field in that region.

The time of travel t (or time of flight) is thus equal to:

$$t = \frac{L}{v} = \boxed{L\sqrt{\frac{m}{2qV_0}}} \tag{21}$$

(b) To estimate the time-resolution of the detector the expression of time of flight is differentiated considering t and m as variables:

$$\partial t = \partial\left(L\sqrt{\frac{m}{2qV_0}}\right) = \frac{L}{\sqrt{2qV_0}} \cdot \partial(\sqrt{m}) = \frac{L}{\sqrt{2qV_0}} \cdot \frac{\partial m}{2\sqrt{m}} \tag{22}$$

$$\partial t = \frac{L}{\sqrt{8qV_0m}}\partial m = \frac{0.5}{\sqrt{1.6 \cdot 10^{-19} \cdot 1000 \cdot 3.95 \cdot 10^{-25}}} \cdot (3.95 \cdot 10^{-25} - 3.90 \cdot 10^{-25}) \approx 110ns \tag{23}$$

The difference in time travel between the two isotopes is in the timescale of $100ns$. Hence, the resolution of the detector has to be better than 10^{-7} seconds in order to distinguish the masses of the two isotopes.