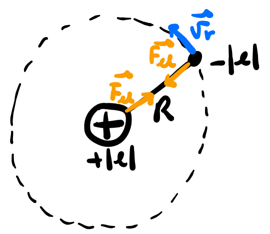
EXERCISE SESSION 2 - SOLUTIONS

Exercise 1)



$R = 5{,}29 \cdot 10^{-11}\ m$

$F_{el} = ?$

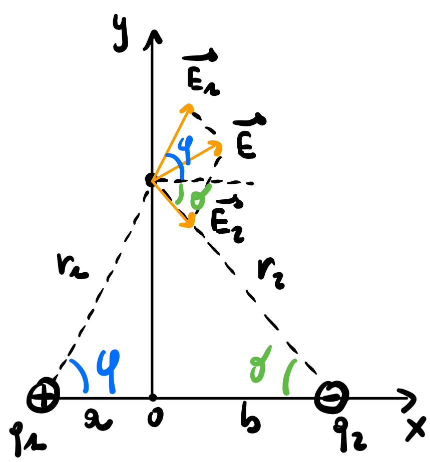
$v_t = ?$

Proton → charge $+|e|$, electron → charge $-|e|$

$$\bullet\ F_{el} = k_e \frac{e^2}{R^2} = 8{,}99 \cdot 10^9 \cdot \frac{(1{,}6 \cdot 10^{-19})^2}{(5{,}29 \cdot 10^{-11})^2} = \boxed{8{,}22 \cdot 10^{-8}\ N}$$

$$\bullet\ F_c = \frac{m_e v_t^2}{R} = F_{el} \rightarrow v_t = \sqrt{\frac{F_{el} \cdot R}{m_e}} = \boxed{2{,}19 \cdot 10^6\ m/s}$$

Exercise 2)



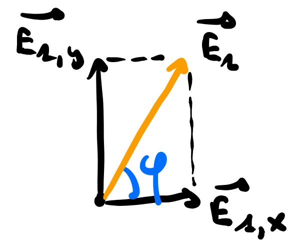
• $\vec{E}$ at $P = (0, y) = ?$

$\vec{E}$ is given by the total contribution of $\vec{E_1}$ and $\vec{E_2}$ → we compute the two different contributions and then we sum them up (SUPERPOSITION)

$\vec{E_1}$ → electric field generated by q_1

$\vec{E_2}$ → electric field generated by q_2

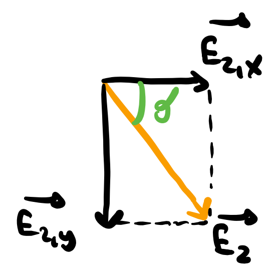
$$|\vec{E_1}| = k_e \frac{|q_1|}{r_1^2} = k_e \frac{|q_1|}{a^2+y^2}$$



$$\rightarrow \vec{E_1} = k_e \frac{|q_1|}{a^2+y^2} \left(\underbrace{\cos\varphi}_{\text{x component}} \hat{x} + \underbrace{\sin\varphi}_{\text{y component}} \hat{y} \right)$$

Both the components are positive because q_1 is positive $\rightarrow \vec{E_1}$ is pointing to larger y values in this configurations $\rightarrow$ the two components (projections) are positive

$$|\vec{E_2}| = k_e \frac{|q_2|}{r_1^2} = k_e \frac{|q_2|}{b^2+y^2}$$



$$\rightarrow \vec{E_2} = k_e \frac{|q_1|}{b^2+y^2} \left(\underbrace{\cos\vartheta}_{\text{x component}} \hat{x} - \underbrace{\sin\vartheta}_{\text{y component}} \hat{y} \right)$$

$\rightarrow$ minus because it is pointing downward

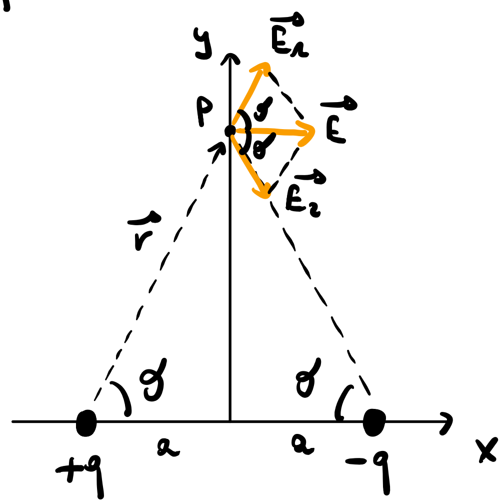
So, the components of the net $\vec{E}$ are:

$$E_x = E_{1,x} + E_{2,x} = \boxed{k_e \frac{|q_1|}{a^2+y^2} \cos\varphi + k_e \frac{|q_2|}{b^2+y^2} \cos\vartheta}$$

$$E_y = E_{1,y} + E_{2,y} = \boxed{k_e \frac{|q_1|}{a^2+y^2} \sin\varphi - k_e \frac{|q_2|}{b^2+y^2} \sin\vartheta}$$

- $\vec{E}(P) = ?$ for $|q_1| = |q_2|$, $a = b$

This particular case is called <u>electric dipole</u>, i.e. two charges of opposite sign where the electric field generated is computed in a point in the space equally spaced from the two charges (such that they result equally spaced with respect to the origin). In this particular case, due to the symmetry, we also have $\varphi = \vartheta$



By using the result got in the previous point, we have:

$$E_x = k_e \frac{q}{a^2+y^2} \cos\varphi + k_e \frac{q}{a^2+y^2} \cos\vartheta = 2k_e \frac{q}{a^2+y^2} \cos\vartheta$$

$$E_y = k_e \frac{q}{b^2+y^2} \cos\varphi - k_e \frac{q}{b^2+y^2} \cos\varphi = 0$$

So, the resulting electric field does not have any y component!

Now, it makes more sense to find an expression for $\cos\vartheta$:

$$\cos\vartheta = \frac{a}{r} = \frac{a}{\sqrt{a^2+y^2}}$$

$$\rightarrow \vec{E} = E_x \hat{x} = 2k_e \frac{q}{a^2+y^2} \frac{a}{\sqrt{a^2+y^2}} \hat{x} = \boxed{k_e \frac{2aq}{\sqrt{(a^2+y^2)^3}} \hat{x}}$$

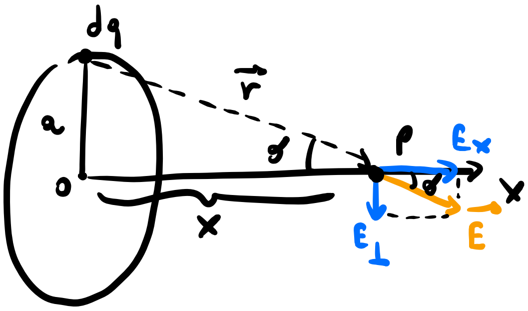
• $y \gg a$

This is a common situation for different application. Basically, we are interested in computing $\vec{E}$ in a point in the space which is much further than the spacing between the two charges. In this case, we can approx: $a^2+y^2 \simeq y^2$, which gives:

$$\vec{E} = E_x \hat{x} \simeq \boxed{k_e \frac{2aq}{y^3}}$$

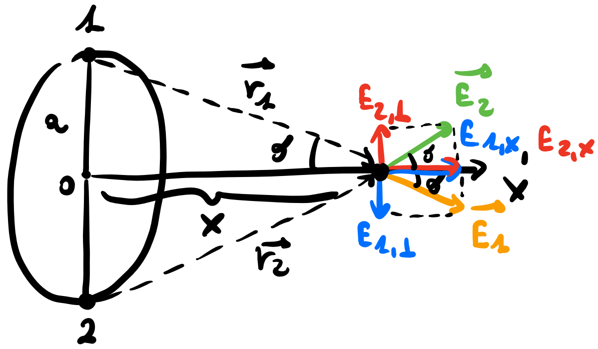
Consideration: even if in computing the $\vec{E}$ created far in the space, we may be tempted in saying that it is zero because the two opposite charges balance them out, this is not true! In fact, this is true only for the y component. However, it is true that $\vec{E}$ decades faster with respect to the field generated by a single charge $\left(\frac{1}{y^3} \text{ vs } \frac{1}{y^2}\right)$, so in the case of the dipole, the field generated goes to zero faster, because of the presence of two charges with opposite sign.

Exercise 3



total charge Q

First observation: symmetry of the system



If we consider two charges (1 and 2 in figure) at the opposite side of the ring, the total field generated by them along the direction perpendicular to the central axis perpendicular to the ring plane ($E_\perp$) is zero, because the two contributions ($E_{1\perp}$ and $E_{2\perp}$ have opposite sign). This consideration is valid for any couple of charges on the ring. So, in the end we are just interested in the x-component

The ring is a continuous of infinitesimal charges dq $\longrightarrow$ we compute the infinitesimal $d\vec{E}$ generated by a single infinitesimal charge dq and then we sum up all the contributions, which means, for a continuous charge distribution, integrating along the ring

$$dE_x = k_e \frac{dq}{r^2} \cos\vartheta = k_e \frac{dq}{a^2+x^2} \cos\vartheta$$

$$\cos\vartheta = \frac{x}{r} = \frac{x}{\sqrt{a^2+x^2}}$$

$$\rightarrow dE_x = k_e \frac{dq}{a^2+x^2} \frac{x}{\sqrt{a^2+x^2}} = \frac{k_e\, x}{\sqrt{(a^2+x^2)^3}}\, dq$$

$$E_x = \int_0^Q dq\, k_e \frac{x}{(a^2+x^2)^{3/2}} = \frac{k_e\, x}{(a^2+x^2)^{3/2}} \underbrace{\int_0^Q dq}_{Q \text{ total charge of the ring}} =$$

$$= \frac{k_e\, x}{(a^2+x^2)^{3/2}}\, Q$$

$$\vec{E} = E_x\, \hat{x} = \boxed{\frac{k_e\, x\, Q}{(a^2+x^2)^{3/2}}}$$

Notice that for $x \gg a$, i.e. far from the ring, as expected the expression becomes:

$$\vec{E} \sim \frac{k_e\, x\, Q}{x^3}\, \hat{x} = k_e \frac{Q}{x^2}\, \hat{x} \quad \text{(point-like)}$$

- Negative charge at the centre of the ring and displaced of $x \ll a$

Field generated by the ring for $x \ll a$:

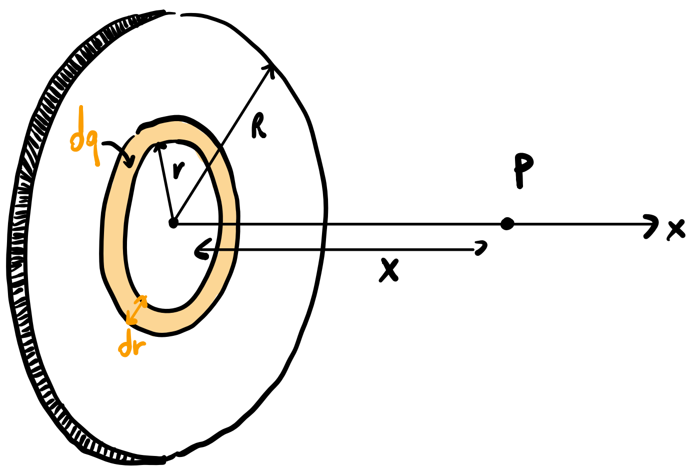
$$E_x = \frac{k_e\, x\, Q}{(a^2+x^2)^{3/2}} \sim \frac{k_e\, x\, Q}{a^3} = \frac{k_e\, Q}{a^3}\, x$$

Force exerted on the negative charge q:

$$\vec{F} = -q \cdot \vec{E} = -q\, E_x = -\frac{k_e\, q\, Q}{a^3}\, x = \boxed{-\text{const} \cdot x}$$

→ Hooke's like force → harmonic motion of the charge

Exercise 4



Charged disk with surface charge density: $\sigma = \frac{Q_{disk}}{A}$

Uniformly charged disk. Idea: considering the disk as a set of concentric rings, so that we can recycle the result of the previous exercise, where we had (neglecting the vector form, since we know there is just the x-component E_x:

$$E_{ring} = \frac{k_e x Q}{(a^2+x^2)^{3/2}}$$

, where Q : total charge of the ring and a radius of the ring

In this case, we have infinitesimal rings with infinitesimal charge : $dq = \sigma \cdot dA = \sigma\, d(\pi r^2) = \sigma\, 2\pi r\, dr$

So, the infinitesimal $\vec{E}$ generated by dq is:

$$dE_x = \frac{k_e x}{(r_1^2+x^2)^{3/2}} \underbrace{(2\pi \sigma r\, dr)}_{dq \text{ (instead of } Q)}$$

↳ r instead of "a" of the previous exercise

In conclusion, to compute the total $\vec{E}$ generated by the disk, we have to integrate over all the disk radius R:

$$E_x = \int_0^R \frac{k_e x}{(r^2+x^2)^{3/2}} 2\pi\sigma r\, dr = 2\pi\sigma k_e x \int dr \frac{r}{(r^2+x^2)^{3/2}} =$$

$$= 2\pi\sigma k_e x \int dr\, r\cdot(r^2+x^2)^{-3/2}$$

Remind: $\int dx\, x^n = \frac{x^{n+1}}{n+1} + C$

$$\int dr\, r\cdot(r^2+x^2)^{-3/2} = \left[\frac{(r^2+x^2)^{-1/2}}{-1/2}\cdot\frac{1}{2}\right]_0^R =$$

$= \frac{1}{2}\cdot 2r$

↳ chain rule : $\int f(g(x))\cdot \underbrace{g'(x)}_{2r}\cdot dx =$
$= F(g(x))$, F primitive of f

$$= -(R^2+x^2)^{-1/2} - \left(-\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{\sqrt{R^2+x^2}}$$

And so, we get:

$$\boxed{E_x = 2\pi\sigma k_e \left(1 - \frac{x}{\sqrt{R^2+x^2}}\right)}$$

• $R \to \infty$, $\vec{E} = ?$

If we assume $x \ll R$ (reasonable, considering that $R \to \infty$), we get:

$$E_x \sim 2\pi\sigma k_e = 2\pi\sigma \frac{1}{4\pi\varepsilon_0} = \boxed{\frac{\sigma}{2\varepsilon_0}}$$

which is the electric field generated by an infinite plane of charges (see appl. of Gauss theorem in the next lecture/exercise)

• Back to R finite radius of the disk

$x \gg R$, $\vec{E} = ?$

$$E_x = 2\pi\sigma k_e \left(1 - \frac{x}{\sqrt{R^2 + x^2}}\right) = 2\pi\sigma k_e \left(1 - \frac{x}{\sqrt{x^2(1 + R^2/x^2)}}\right) =$$

$$= 2\pi\sigma k_e \left(1 - \frac{1}{\sqrt{1 + R^2/x^2}}\right)$$

We use: $\sqrt{1 + \frac{R^2}{x^2}} \simeq 1 + \frac{R^2}{2x^2}$ for $x \gg R$ $\left(\to \frac{R^2}{x^2} \ll 1\right)$

$$\to E_x \simeq 2\pi\sigma k_e \left(1 - \frac{2x^2}{2x^2 + R^2}\right) = 2\pi\sigma k_e \frac{R^2}{2x^2 + R^2} \simeq$$

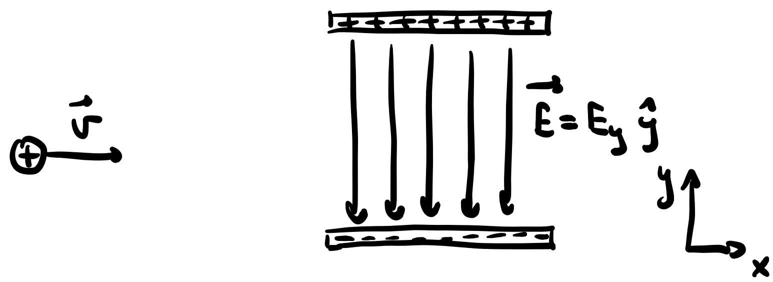
$$\simeq \cancel{2}\pi\sigma k_e \frac{R^2}{\cancel{2}x^2} \quad (x \gg R)$$

Remind: $\sigma = \frac{Q}{A} = \frac{Q}{\pi R^2}$

$$\rightarrow E_x \approx \cancel{\pi}\frac{Q}{\cancel{\pi}\cancel{R^2}} k_e \frac{\cancel{R^2}}{x^2} = \boxed{k_e \frac{Q}{x^2}}$$

which is the electric field generated by a point-like charge, because the disk at long distance looks like a single charge Q

Exercise 5



$\vec{v}_i = (4.50 \cdot 10^5)\hat{x}$ m/s

$\vec{E} = (3.60 \cdot 10^3)\hat{y} \frac{N}{C}$

- $d = 5.00$ cm $\rightarrow t = ?$

In the orizontal direction the motion is uniform, i.e. at constant speed because the $\vec{E}$ field is directed only in the y direction!

$$d = v_x \cdot t \rightarrow t = \frac{d}{v_x} = \frac{5 \cdot 10^{-2}}{4.5 \cdot 10^5} = \boxed{111 \text{ ms}}$$

- $y_f = ?$ after $t = 111$ ms

y-direction $\rightarrow$ constant field $\rightarrow$ constant force $\rightarrow$ constant a_y

$$a_y = \frac{F}{m} = \frac{qE}{m} = \frac{1{,}6\cdot 10^{-19} \cdot 9{,}6\cdot 10^{3}}{1{,}67\cdot 10^{-27}} = 9{,}21\cdot 10^{11}\ m/s^2$$

Constant accel. $\rightarrow$ $y = \frac{1}{2} a_y t^2 + \cancel{v_{y,i}}^{\,0} \cdot t = \frac{1}{2} a_y \cdot t^2 =$

$$= \frac{1}{2} \cdot 9{,}2\cdot 10^{11} \cdot (1{,}11\cdot 10^{-7})^2 = 5{,}68\cdot 10^{-3}\ m = \boxed{5{,}68\ mm}$$

- $v_{x,f} = ?\ v_{y,f} = ?$ after $d = 5.00\ cm$

$v_{x,f} = v_{x,i} = 4{,}50\cdot 10^{5}\ m/s$

$v_{y,f} = \cancel{v_{y,i}}^{\,0} + a_y t = 9{,}21\cdot 10^{11} \cdot 1{,}11\cdot 10^{-7} = 1{,}02\cdot 10^{5}\ m/s$

$$\boxed{\vec{v}_f = (4{,}50\cdot 10^{5}\ \hat{x} + 1{,}02\cdot 10^{5}\ \hat{y})\ m/s}$$