EXERCISE SESSION 1 – SOLUTIONS

Exercise 1

1) $\vec{A}\cdot\vec{B} = ?$ $\vec{A}\times\vec{B} = ?$

a) $\vec{A} = (6,2,1)$, $\vec{B} = (8,9,2)$

$$\vec{A}\cdot\vec{B} = (6\hat{x}+2\hat{y}+1\hat{z})(8\hat{x}+9\hat{y}+2\hat{z}) =$$

$$= 6\cdot 8\,\underbrace{\hat{x}\cdot\hat{x}}_{1} + 6\cdot 9\,\underbrace{\hat{x}\cdot\hat{y}}_{0} + 6\cdot 2\cdot\underbrace{\hat{x}\cdot\hat{z}}_{0} + 2\cdot 8\cdot\underbrace{\hat{y}\cdot\hat{x}}_{0} + 2\cdot 9\cdot\underbrace{\hat{y}\cdot\hat{y}}_{1} +$$

$$+ 2\cdot 2\,\underbrace{\hat{y}\cdot\hat{z}}_{0} + 1\cdot 8\cdot\underbrace{\hat{z}\cdot\hat{x}}_{0} + 1\cdot 9\cdot\underbrace{\hat{z}\cdot\hat{y}}_{0} + 1\cdot 2\cdot\underbrace{\hat{z}\cdot\hat{z}}_{1} =$$

$= 48 + 18 + 2 = \boxed{68}$ (the result is a <u>scalar</u>, i.e. a <u>number</u>)

$$\vec{A}\times\vec{B} = (6\hat{x}+2\hat{y}+1\hat{z})\times(8\hat{x}+9\hat{y}+2\hat{z})$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y) - \hat{y}(A_xB_z - A_zB_x) + \hat{z}(A_xB_y - A_yB_x)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 6 & 2 & 1 \\ 8 & 9 & 2 \end{vmatrix} = \hat{x}(2\cdot 2 - 1\cdot 9) - \hat{y}(6\cdot 2 - 1\cdot 8) + \hat{z}(6\cdot 9 - 2\cdot 8) =$$

$$= -5\hat{x} - 4\hat{y} + 38\hat{z} = \boxed{(-5,-4,+38)}$$

$\rightarrow$ the result is a <u>vector</u>

b) $\vec{A} = (8,1,7)$, $\vec{B} = (9,6,9)$

$\vec{A}\cdot\vec{B} = 8\cdot 9 + 1\cdot 6 + 7\cdot 9 = \boxed{141}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 8 & 1 & 7 \\ 9 & 6 & 9 \end{vmatrix} = (9-42, -(72-63), 48-9) =$$

$$= \boxed{(-33, -9, 39)}$$

c) $\vec{A} = (5, 2, 5)$, $\vec{B} = (-10, -4, -10)$

$\vec{A} \cdot \vec{B} = -50 - 8 - 50 = \boxed{-108}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 5 & 2 & 5 \\ -10 & -4 & -10 \end{vmatrix} = (-20 + 20, -(-50+50), -20+20) = \boxed{(0, 0, 0)} \rightarrow \vec{A}, \vec{B} \text{ are } \underline{\text{parallel}}$$

d) $\vec{A} = (-3, 8, 2)$, $\vec{B} = (0, -8, 1)$

$\vec{A} \cdot \vec{B} = 0 - 64 + 2 = \boxed{-62}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3 & 8 & 2 \\ 0 & -8 & 1 \end{vmatrix} = (24-0, -(-3-0), 24-0) = \boxed{(24, 3, 24)}$$

2) $\vec{A} = (3, 3, 3)$, $\vec{B} = (6, 1, 7)$

$\vec{r} = ?$ separation vector between A and B

$|\vec{r}| = ?$ $\hat{r} = ?$

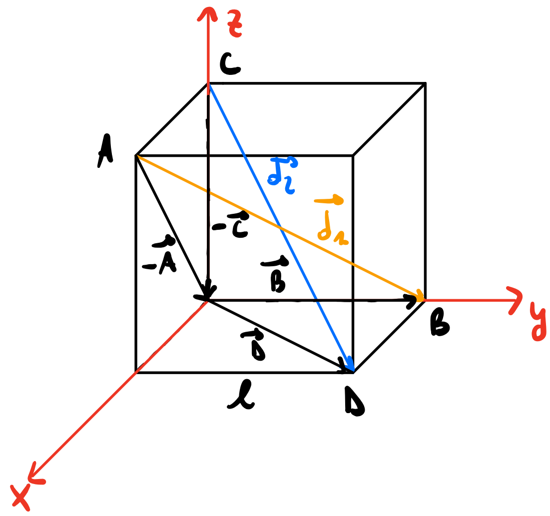
$\vec{r} = \vec{B} - \vec{A} = (6-3, 1-3, 7-3) = \boxed{(-3, -2, 4)}$

$|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}} = \sqrt{(-3)^2 + (-2)^2 + 4^2} = \sqrt{9+4+16} = \boxed{\sqrt{23}}$

Since $\vec{r} = |\vec{r}| \cdot \hat{r} \longrightarrow \hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\boxed{\hat{r} = \left(-\frac{3}{\sqrt{23}}, -\frac{2}{\sqrt{23}}, \frac{4}{\sqrt{23}}\right)}$$

3)



Let's consider $\vec{d}_1$ and $\vec{d}_2$ (notice that there are 4 in total):

$\vec{d}_1 = \vec{B} - \vec{A} = (0,\ell,0) + -(\ell,0,\ell) = \underline{(-\ell,\ell,-\ell)}$

$\vec{d}_2 = \vec{D} - \vec{C} = (\ell,\ell,0) + -(0,0,\ell) = \underline{(\ell,\ell,-\ell)}$

Def. of scalar product: $\vec{d}_1 \cdot \vec{d}_2 = |\vec{d}_1| \cdot |\vec{d}_2| \cdot \cos\vartheta$ ↳ angle between the 2 vectors

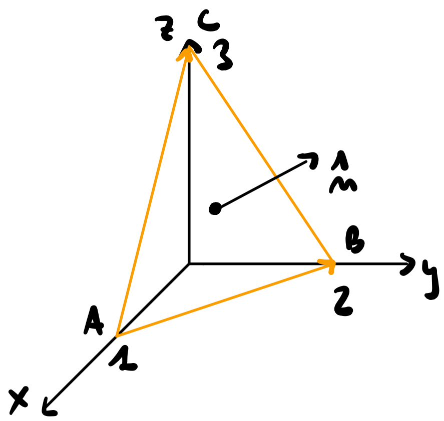
$$\rightarrow \vartheta = \arccos\left(\frac{\vec{d}_1 \cdot \vec{d}_2}{|\vec{d}_1||\vec{d}_2|}\right)$$

$$\vec{d}_1 \cdot \vec{d}_2 = -\ell\cdot\ell + \ell\cdot\ell + (-\ell)\cdot(-\ell) = \ell^2$$

$$|\vec{d}_1| = |\vec{d}_2| = \sqrt{\ell^2+\ell^2+\ell^2} = \sqrt{3}\,\ell$$

$$\rightarrow \vartheta = \arccos\left(\frac{\ell^2}{\sqrt{3}\ell\cdot\sqrt{3}\ell}\right) = \boxed{\arccos\left(\frac{1}{3}\right)}$$

4)



$\hat{n} = ?$ ($\perp$ to the orange plane)

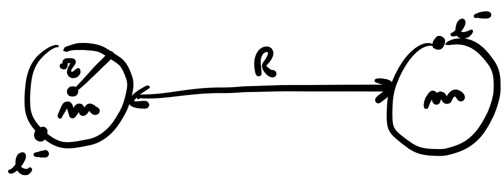
$\vec{AB} = \vec{B} - \vec{A} = (0,2,0) - (1,0,0) =$
$= (-1,2,0)$

$\vec{AC} = \vec{C} - \vec{A} = (0,0,3) - (1,0,0) =$
$= (-1,0,3)$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (6, -(-3), -(-2)) = (6, 3, 2)$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{(6,3,2)}{\sqrt{6^2+3^2+2^2}} = \frac{(6,3,2)}{7} = \boxed{\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)}$$

Exercise 2



$k_e = 9 \cdot 10^9 \ \frac{N \cdot m^2}{C^2}$

$e = 1{,}6 \cdot 10^{-19} \ C$

$G = 6{,}7 \cdot 10^{-11} \ \frac{m^3}{kg \cdot s^2}$

$\rho = 1 \cdot 10^3 \ kg/m^3$

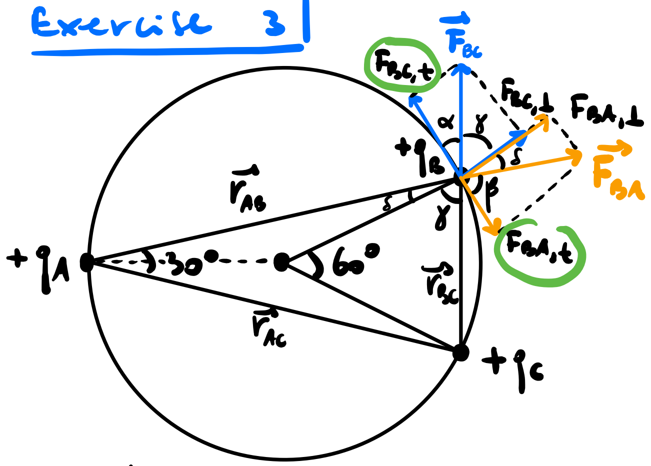
$r = ?$

$$|\vec{F}_{el}| = |\vec{F}_g| \longrightarrow k \frac{e \cdot e}{\cancel{R^2}} = G \frac{m \cdot m}{\cancel{R^2}}$$

$$m = \sqrt{\frac{ke^2}{G}} = \rho V = \rho \frac{4}{3} \pi r^3$$

$$\longrightarrow r^3 = \frac{3}{4\pi\rho} \sqrt{\frac{ke^2}{G}} \longrightarrow r = \sqrt[3]{\frac{3}{4\pi\rho}} \sqrt[6]{\frac{ke^2}{G}} = \boxed{\sqrt[6]{\frac{9ke^2}{16\pi^2\rho^2 G}}}$$

Exercise 3



@ equilibrium :

$F_{BC,t} = F_{BA,t}$ → force acting on B due to A, tangent component

↳ force acting on B due to C, tangent component

$$\vec{F}_{BC} \cdot \cos\alpha = \vec{F}_{BA} \cdot \cos\beta$$

$$\cancel{\frac{1}{4\pi\varepsilon_0}} \frac{\cancel{q_B} \cdot q_C}{|\vec{r}_{BC}|^2} \cos\alpha = \cancel{\frac{1}{4\pi\varepsilon_0}} \frac{\cancel{q_B}\, q_A}{|\vec{r}_{AB}|^2} \cos\beta$$

$$\frac{q_A}{q_C} = \frac{q_A}{q_B} = \frac{|\vec{r}_{AB}|^2}{|\vec{r}_{BC}|^2} \frac{\cos\alpha}{\cos\beta} \qquad (\text{since } q_B = q_C)$$

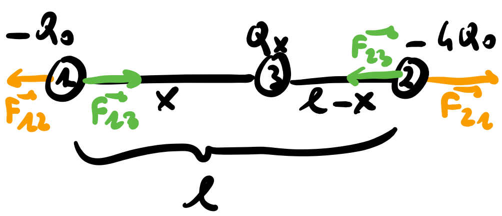
$$\alpha = 90° - \gamma = 90° - 60° = 30°$$

$$\beta = 90° - \delta = 90° - \frac{1}{2} \cdot 30° = 75°$$

$$\frac{1}{2} |\vec{r}_{BC}| = |\vec{r}_{AB}| \cos(\gamma + \delta) \longrightarrow |\vec{r}_{BC}| = 2\,|\vec{r}_{AB}| \cos(60° + 15°)$$

$$\longrightarrow \frac{q_A}{q_B} = \frac{\cancel{|\vec{r}_{AB}|^2}}{4\cancel{|\vec{r}_{AB}|^2} \cdot \cos^2 75°} \cdot \frac{\cos 30°}{\cos 75°} = \boxed{\frac{\cos 30°}{4\cos^3 75°}}$$

Exercise 4



$|Q_x|$ and position such that the system is in equilibrium ?

$$\underset{A)}{|\vec{F}_{12}| = |\vec{F}_{13}|}, \quad \underset{B)}{|\vec{F}_{21}| = |\vec{F}_{23}|}$$

A) $\left| \cancel{k}\, \frac{-\cancel{q_0}\cdot(-4q_0)}{\ell^2} \right| = \left| \cancel{k}\, \frac{-\cancel{q_0}\cdot Q_x}{x^2} \right| \to Q_x = \frac{4x^2}{\ell^2} q_0$ ①

B) $\left| \cancel{k}\, \frac{-q_0(-\cancel{4q_0})}{\ell^2} \right| = \left| \cancel{k}\, \frac{-\cancel{4q_0}\cdot Q_x}{(\ell-x)^2} \right| \to Q_x = \frac{(\ell-x)^2}{\ell^2} q_0$ ②

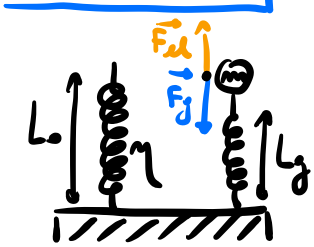
① = ② $\to 4x^2 = (\ell - x)^2 \to 4x^2 = \ell^2 + x^2 - 2\ell x$

$\to 3x^2 + 2\ell x - \ell^2 = 0$

$\to x = \frac{-\ell \pm \sqrt{\ell^2 + 3\ell^2}}{3} = \frac{-\ell \pm 2\ell}{3}$ $\to \boxed{\ell/3}$ ✓ ; $\to -\ell$ (not accept.)

$\to Q_x = \frac{4/9\, \cancel{\ell^2}}{\cancel{\ell^2}} q_0 = \boxed{\frac{4}{9} q_0}$

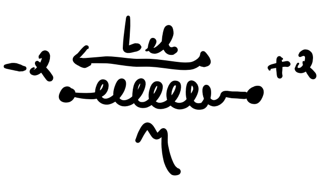
Exercise 5



First part:

$|\vec{F_g}| = |\vec{F_{el}}| \to mg = -\eta\, \Delta x = -\eta(L_g - L_0)$

$\to \eta = \frac{mg}{L_0 - L_g}$ ①



Second part:

$Q = ?$ when $L = L_{el}$

$|\vec{F_{coul}}| = |\vec{F_{el}}| \to k \frac{Q^2}{L_{el}^2} = -\eta(L_{el} - L_0)$

$Q^2 = \frac{\eta(L_0 - L_{el})\cdot L_{el}^2}{k} \overset{①}{=} \frac{mg}{L_0 - L_g} \frac{(L_0 - L_{el})\, L_{el}^2}{k}$

$\to \boxed{Q = \sqrt{\frac{mg(L_0 - L_{el})}{k(L_0 - L_g)}}\, L_{el}}$