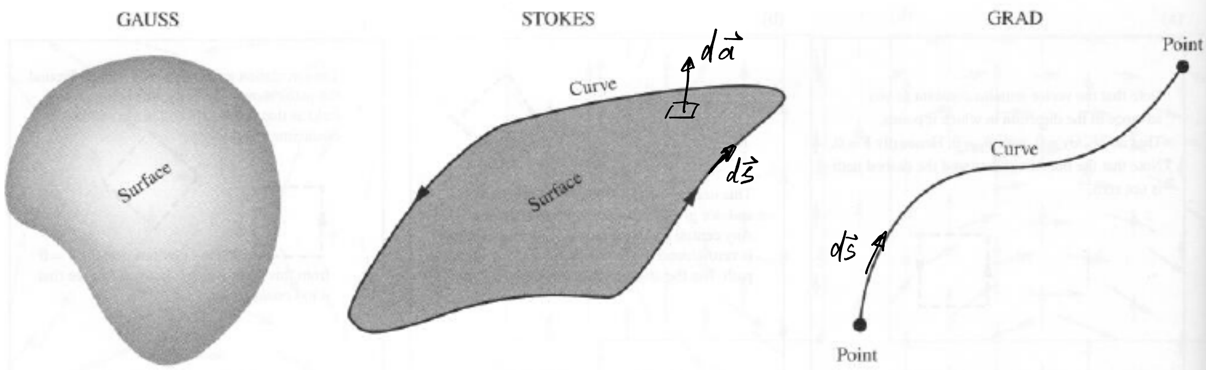
Short summary on: divergence, curl, and gradient (cartesian coordinates)



Surface encloses volume | Curve encloses surface | Points enclose curve

$$\int_{\text{surface}} \mathbf{F}\cdot d\mathbf{a} = \int_{\text{volume}} \operatorname{div}\mathbf{F}\, dV$$

$$\int_{\text{curve}} \mathbf{A}\cdot d\mathbf{s} = \int_{\text{surface}} \operatorname{curl}\mathbf{A}\cdot d\mathbf{a}$$

$$\phi_2 - \phi_1 = \int_{\text{curve}} \operatorname{grad}\phi\cdot d\mathbf{s}$$

In Cartesian coordinates, with $\vec{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$:

$$\operatorname{div}\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \vec{\nabla}\cdot\mathbf{F}$$

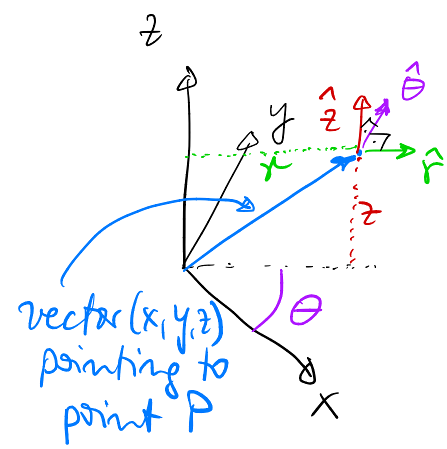
$$\operatorname{curl}\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \vec{\nabla}\times\mathbf{A}$$

$$\operatorname{grad}\phi = \hat{\mathbf{x}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial \phi}{\partial z} = \vec{\nabla}\phi$$

Note: Following a convention, bold letters indicate vectors.

Div, grad, curl in different coordinate systems

Cylindrical (polar) coordinates



$$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{z}}$$

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right)\hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical coordinates

r: radial coordinate; θ: polar angle; φ: azimuthal angle

$$ds = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

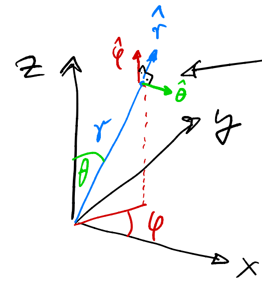
$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\varphi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left(\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(rA_\varphi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\varphi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

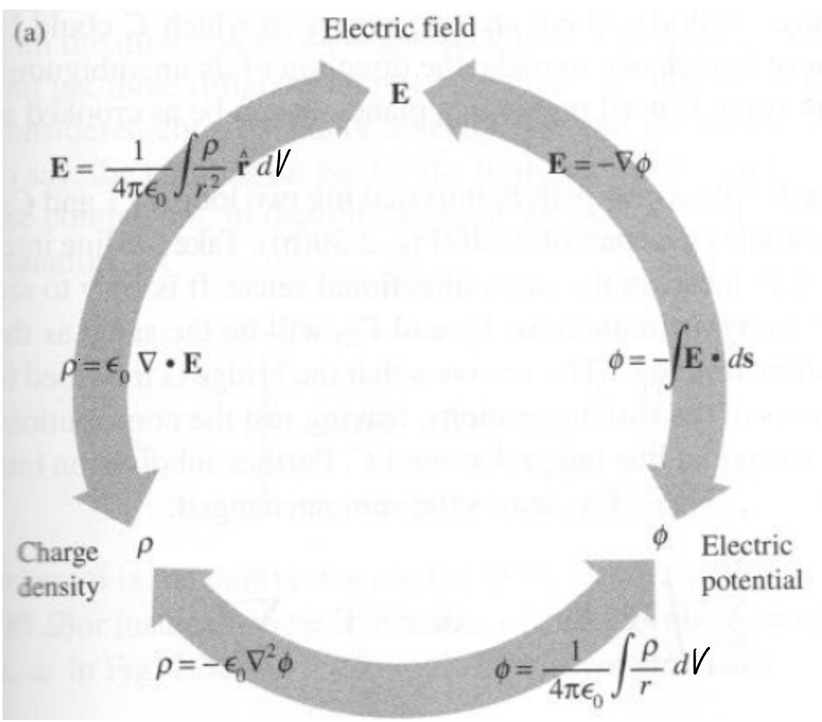


three orthogonal unit vectors

This is a convention in physics (different from maths)

(from Purcell)

How div, curl and grad are exploited in electrostatics:



taken from Purcell

taken from Purcell

dV: volume integral

Note that in the "compact" formula given in this "summary sketch" of Purcell the distance and unit vectors in the Coulomb law must be extended to the version used in the lecture.

Purcell uses $d\vec{s}$ for $d\vec{l}$ used in the lecture.

Further details on the individual operations and examples

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Relevant Theorems

Divergence theorem

$$\int_{\text{surface}} \mathbf{F} \cdot d\mathbf{a} = \int_{\text{volume}} \operatorname{div} \mathbf{F} \, dv$$

Stokes' theorem

$$\int_{\text{curve}} \mathbf{A} \cdot d\mathbf{s} = \int_{\text{surface}} \operatorname{curl} \mathbf{A} \cdot d\mathbf{a}$$

Gradient theorem

$$\phi_2 - \phi_1 = \int_{\text{curve}} \operatorname{grad} \phi \cdot d\mathbf{s}$$

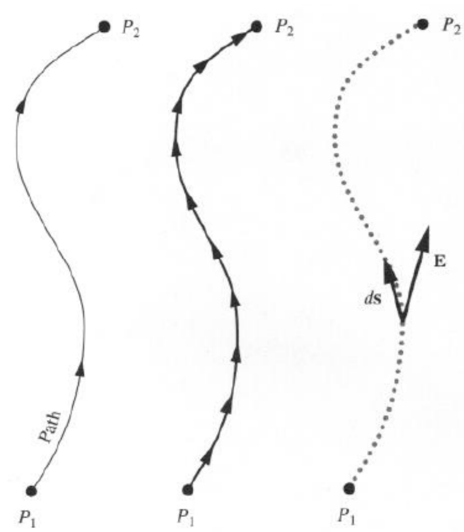


Figure 2.1.
Showing the division of the path into path elements $d\mathbf{s}$.

Gradient

We remind the reader that a partial derivative with respect to x, of a function of x, y, z, written simply $\partial f/\partial x$, means the rate of change of the function with respect to x with the other variables y and z held constant. More precisely,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}. \quad (2.11$$

As an example, if $f = x^2yz^3$,

$$\frac{\partial f}{\partial x} = 2xyz^3, \quad \frac{\partial f}{\partial y} = x^2z^3, \quad \frac{\partial f}{\partial z} = 3x^2yz^2. \quad (2.12$$

"grad f," or ∇f:

$$\boxed{\nabla f \equiv \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}}$$

Operator "gradient"

Example

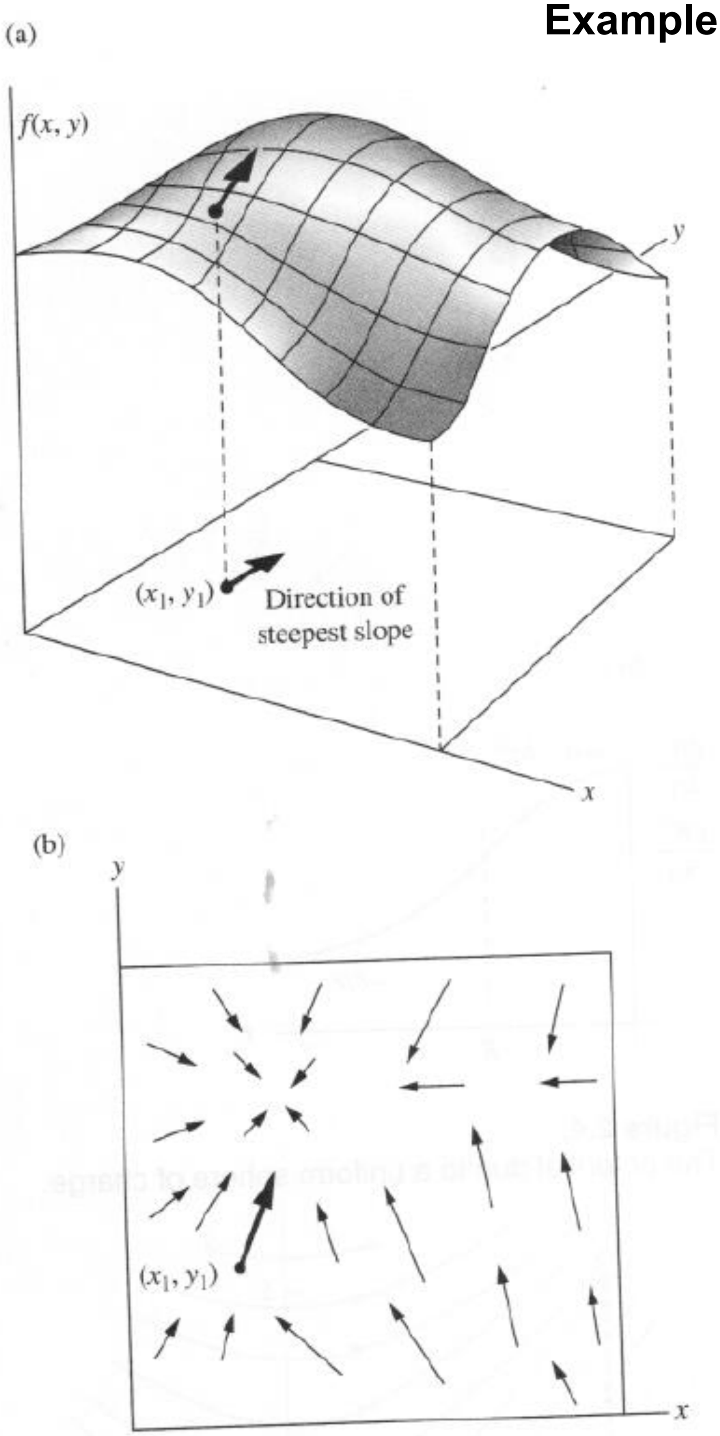


Figure 2.5.
The scalar function $f(x, y)$ is represented by the surface in (a). The arrows in (b) represent the vector function, grad f.

taken from:
Purcell

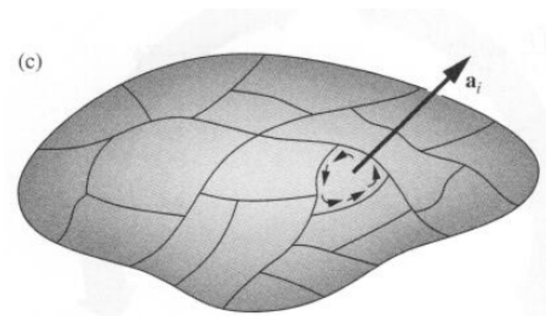
Curl of a vector function

$$(\text{curl } \vec{F})\cdot\hat{n} = \lim_{a_i \to 0} \frac{\int_{C_i} \vec{F}\cdot d\vec{s}}{a_i}$$

circulation

area that is "circulated"

curl : – measure of the line integral of a vector field around a small closed curve



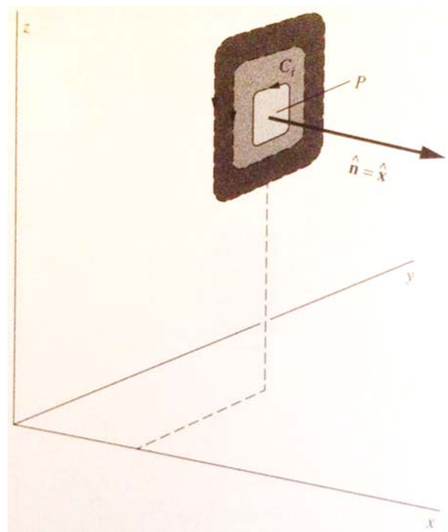


Figure 2.32.
The patch shrinks around P, keeping its normal pointing in the x direction.

- curl $\vec{F}$ is a vector function of coordinates.

- $$\text{curl } \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

  formula in the form of a determinant

  $= \vec{\nabla} \times \vec{F}$ ("del cross F" = "curl F")

• The static electric field is conservative
$\Rightarrow$ curl is zero!

Divergence

Assume a vector function $\vec{F}(x,y,z)$:

$$\operatorname{div} \vec{F} = \lim_{V_i \to 0} \frac{1}{V_i} \int_{S_i} \vec{F} \cdot d\vec{a}_i$$

V_i : volume including the point (x,y,z) in question
S_i : surface for the integral to be taken over (surface of V_i)
$\vec{a}_i$: normal vector on surface S_i

- $\operatorname{div} \vec{F}$ is the flux out of V_i, per unit of volume, in the limit of infinitesimal V_i
- scalar quantity
- it may vary from place to place (x,y,z)

$\Rightarrow$ scalar function of the coordinates

If one uses this form, one can arrive at Gauss's theorem:

$$\int_S \vec{F} \cdot d\vec{a} = \sum_{i=1}^{N} \int_{S_i} \vec{F} \cdot d\vec{a}_i = \sum_{i=1}^{N} V_i \left[\frac{\int_{S_i} \vec{F} \cdot d\vec{a}_i}{V_i} \right]$$

In the limit of $N \to \infty$ and $V_i \to 0$, one gets the divergence on the right side and the sum transforms into a volume integral:

$$\underbrace{\boxed{\int_S \vec{F}\, d\vec{a} = \int_V \operatorname{div} \vec{F}\, dV \text{ volume}}}$$

This is Gauss' theorem or divergence theorem

- It holds for any vector field for which the limit $\operatorname{div} \vec{F}$ exists.

- $$\operatorname{div} \vec{F} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

in compact form $\vec{\nabla} \cdot \vec{F}$ ($\vec{\nabla}$: gradient operator)

$$\boxed{\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}}$$ Operator "divergence"

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$$

$\boxed{\text{divergence :- measure of flux of a vector field out of a small volume}}$

- $\operatorname{div}\vec{F} > 0$: net "outflow" in the neighborhood of some point
- $\operatorname{div}\vec{F} > 0$: possible even if one or two of the three partial derivatives are negative
- $\operatorname{div}\vec{F}$ expresses only one aspect of the spatial variation of a vector field

Laplacian (or divergence of the gradient)

Define a potential function $\phi(x,y,z)$. Then:

$$\vec{\mathcal{E}} = -\operatorname{grad}\phi = -\left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right)$$

$$\operatorname{div}\vec{\mathcal{E}} = \frac{\partial\mathcal{E}_x}{\partial x} + \frac{\partial\mathcal{E}_y}{\partial y} + \frac{\partial\mathcal{E}_z}{\partial z}$$

From the comparison one gets:

$$\mathcal{E}_x = -\frac{\partial\phi}{\partial x}$$

Combining this for all components:

$$\boxed{\operatorname{div}\vec{E} = -\underbrace{\operatorname{div}\,\operatorname{grad}}_{\nabla^2}\phi = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)}$$

∇^2 : Laplacian operator or the Laplacian (also Δ)

$$\boxed{\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

Operator "Laplacian"