1 Line Integral of a Scalar Field

1.1 References

Physics course: Fields (Electric field Component, Magnetic Field Component).

Analysis III course: definition 2.1 (i) (p. 15). example 2.3 (i) (p. 16). exercise 2.4 (p. 18).

1.2 Comparative example

$$\text{Function : } f : \mathbb{R}^3 \to \mathbb{R} : f(x,y,z) = 1$$

$$\text{Curve } \Gamma : \gamma : [0;\pi] \to \mathbb{R}^3 : \gamma(t) = \left(\frac{\sqrt{2}}{2} r \sin t, \frac{\sqrt{2}}{2} r \sin t, r \cos t\right)$$

1.2.1 Analysis' view

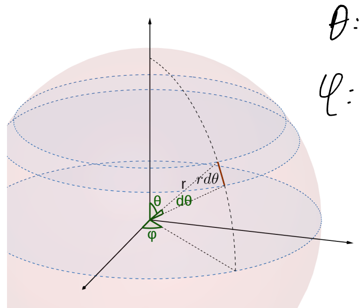
$$\int_\Gamma f \; dl$$

$$\gamma'(t) = \left(\frac{\sqrt{2}}{2} r \cos t, \frac{\sqrt{2}}{2} r \cos t, -r \sin t \right)$$

$$||\gamma'(t)|| = \boxed{r}$$

$$\int_\Gamma f \; dl = \int_0^\pi f(\lambda(t)) \, ||\gamma'(t)|| \; dt$$

1.2.2 Physics' view



θ: polar angle

φ: azimuthal angle

$$\int_\Gamma f \; dl$$

Polar line element:

$$dl = \boxed{r} \; d\boldsymbol{\theta}$$

$$\Rightarrow \int_\Gamma f \; dl = r \int_0^\pi d\boldsymbol{\theta} = \pi r$$

See remark, iii (p. 16) for sign consistency.

Check situation's physics for sign consistency.

2 Line Integral of a Vector Field

2.1 References

Physics course: Energy (Work, Potential).

Analysis III course: definition 2.1 (ii) (p. 15). example 2.3 (ii) (p. 16). exercises 2.2, 2.3 (p. 17), 2.6 (p. 18).

2.2 Comparative example

2.2.1 Analysis' view

$$\int_\Gamma F \cdot dl = \int_\Gamma (F \cdot \tau) \; dl$$

2.2.2 Physics' view

$$\int_\Gamma \boldsymbol{F} \; \boldsymbol{dl} = \int_\Gamma (\boldsymbol{F} \cdot \boldsymbol{\tau}) \; dl$$

τ and $\boldsymbol{\tau}$ being the unit vector tangential to the curve at the functions evalution point. (Direction of integration.)

Comment : A line integral of a vector field can be computed as a line integral of a scalar field. However, see the Analysis III course definition (mentioned above) for a more formal way of performing this computation.

See remark, iii (p. 16) for sign consistency.

Check situation's physics for sign consistency.

3 Surface Integral of a Scalar Field

3.1 References

Physics course: Fields (Electric field Component, Magnetic Field Component). Buoyancy (Component), Newton's law for a fluid (Component).

Analysis III course: definition 5.2 (p. 54). example 5.5 (p. 55). exercise 5.1 (p. 56).

3.2 Comparative example

$$\text{Function}: f : \mathbb{R}^3 \to \mathbb{R} : f(x, y, z) = 1$$

$$\text{Surface } \Sigma : \sigma : [0; \pi] \times [0; 2\pi] \to \mathbb{R}^3 : \sigma(u, v) = (r \sin u \cos v, r \sin u \sin v, r \cos u)$$

3.2.1 Analysis' view

$$\iint_\Sigma f \ ds$$

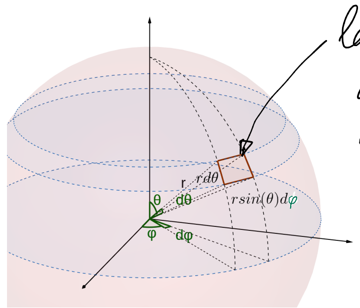
$$\frac{\partial \sigma}{\partial u}(u, v) = (r \cos u \cos v, r \cos u \sin v, -r \sin u)$$

$$\frac{\partial \sigma}{\partial v}(u, v) = (-r \sin u \sin v, r \sin u \cos v, 0)$$

$$\left| \frac{\partial \sigma}{\partial u}(u, v) \wedge \frac{\partial \sigma}{\partial v}(u, v) \right| = \boxed{r^2 \sin u}$$

$$\iint_\Sigma f \ ds = \iint_A f(\sigma(u, v)) \left| \frac{\partial \sigma}{\partial u}(u, v) \wedge \frac{\partial \sigma}{\partial v}(u, v) \right| \ du \ dv$$

3.2.2 Physics' view



labelled often da (sometimes ds if confusion arises otherwise)

$$\iint_\Sigma f \ ds \quad \text{or} \quad \iint_{\text{surface}} f \, da$$

Spherical surface element:

$$ds = \boxed{r^2 \sin \theta} \ d\theta \ d\varphi$$

$$\Rightarrow \iint_\Sigma f \ ds = r^2 \int_0^{2\pi} \int_0^{\pi} \sin u \ du \ dv = 2r^2 \int_0^{2\pi} \ dv = 4\pi r^2$$

See remark, iii (p. 55) for sign consistency. *Check situation's physics for sign consistency.*

4 Surface Integral of a Vector Field

4.1 References

Physics course: Fluxes (Electric Flux, Magnetic Flux, Electric Current), Gauss' Theorem, Magnetic Flux, Maxwell's magnetic monopole equation Energy (Poyinting vector integration). Flows (Mass flow, Volumetric flow rate).

Analysis III course: definition 5.3 (p. 54). example 5.6 (p. 56). exercises 5.2 (p. 56), 5.4 (p. 57).

4.2 Comparative example

4.2.1 Analysis' view

$$\iint_\Sigma F \cdot ds = \iint_\Sigma (F \cdot \nu) \ ds$$

4.2.2 Physics' view

$$\iint_\Sigma \boldsymbol{F} \ d\boldsymbol{a} = \iint_\Sigma (\boldsymbol{F} \cdot \boldsymbol{\nu}) \ ds$$

ν and $\boldsymbol{\nu}$ being the unit vector normal to the surface at the functions evalution point. (Direction arbitrarly chosen.)

Comment : A surface integral of a vector field can be computed as a surface integral of a scalar field. However, see the Analysis III course definition (mentioned above) for a more formal way of performing this computation.

See remark, iii (p. 55) for sign consistency. *Check situation's physics for sign consistency.*

5 Volume Integral of a Scalar Field

5.1 References

Physics course: Fields (Electric field Component, Magnetic Field Component). Newton's law for a fluid (Component).

Analysis II course: definition §14.6.2 (p. 396). exercises §14.7.1 (p. 401), §14.7.2, §14.7.3 (p. 402).

5.2 Comparative example

$$\text{Function : } f : \mathbb{R}^3 \to \mathbb{R} : f(x,y,z) = 1$$

$$\text{Domain : } D = B_r^3(0,0,0) : C = [0;r] \times [0;2\pi] \times [0;\pi]$$

$$\Phi : C \to D : \Phi(\rho,\theta,\phi) = (\rho \sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)$$

5.2.1 Analysis' view

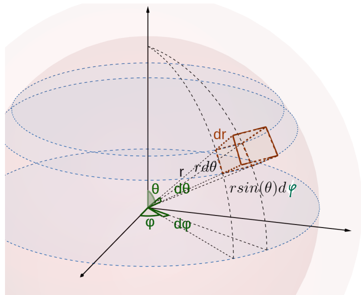
$$\iiint_D f(x,y,z)\ dx\ dy\ dz$$

$$D_\Phi = \begin{bmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{bmatrix}$$

$$|D_\Phi| = \boxed{\rho^2 \sin\theta}$$

$$\iiint_D f(x,y,z)\ dx\ dy\ dz = \iiint_C f(\rho,\theta,\phi)\ |D_\Phi|\ d\rho\ d\theta\ d\phi$$

5.2.2 Physics' view



$$\iiint_D f(x,y,z)\ dV$$

Spherical volume element:

$$dV = dx\ dy\ dz = \boxed{r^2 \sin\theta}\ dr\ d\theta\ d\varphi \quad (1)$$

$$\Rightarrow \iiint_D f(x,y,z)\ dx\ dy\ dz = \iiint_C f(r,\theta,\varphi)\ r^2 \sin\theta\ dr\ d\theta\ d\varphi$$

$$V_{Sphere} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2 \sin\theta\ dr\ d\theta\ d\varphi = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3} r^3 \sin\theta\ d\theta\ d\varphi = \int_0^{2\pi} \frac{2}{3} r^3\ d\varphi = \frac{4}{3}\pi r^3$$

The example f=1 provides the volume of a sphere.

6 References

Analysis III course refers to Dacorogna, Bernard, and Tanteri, Chiara. *Analyse Avancée Pour Ingénieurs.* Quatrième édition ed. Enseignement Des Mathématiques. Lausanne: Presses Polytechniques Et Universitaires Romandes, 2018.

Analysis II course refers to Douchet, Jacques, and Zwahlen, Bruno. *Calcul Différentiel Et Intégral : Tronc Commun, Analyse I, 2016-2017.* Nouvelle édition Revue Et Augmentée De L'ouvrage] ed. Enseignement Des Mathématiques. Lausanne: Presses Polytechniques Et Universitaires Romandes, 2016.

(1) In physics we often stay with coordinate r here, because the letter ρ could be confused with density.