

Exercise sheet 9: Induction law, generator, power

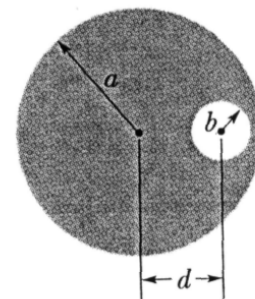
13/11/2024

We indicate the challenges of the problems by categories I (“warming-up”), II (“exam-level”), III (“advanced”). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.

(Cavity in a wire/Category II/After training: 30 min)

(I) What is the magnetic field $\vec{B}$ inside a cylindrical conducting wire carrying a uniform current density j , at a distance r away from the axis? Use Ampere’s law.
 (II) The figure shows the section of a long cylindrical conducting material of radius a , within which is a cylindrical cavity of radius b . The cylinders have parallel axes separated by a distance d , see the figure. A current i is uniformly distributed within the conducting material (in dark grey in the figure). Answer the following questions.
 Hint: Use the result of the preliminary question (I) and the superposition principle.



- Calculate the magnetic field along the axis of the cavity.
- Show that the magnetic field is constant in the cavity.

Exercise 2.

(Infinite Solenoid/Category II)

Consider an infinitely long solenoid with n -turns per unit length. The long axis is along the $+\vec{z}$ -direction. The current I flows along the $\vec{e}_\varphi$ direction in cylindrical coordinates. We assume the solution developed in the lecture concerning the uniform magnetic field $\vec{B}$ inside the solenoid:

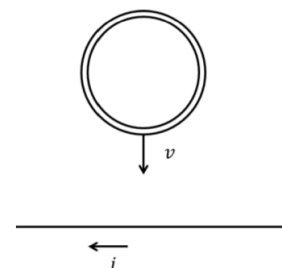
$$\vec{B} = \mu_0 n I \vec{e}_z \quad (1)$$

Use the formula 1, the definition for magnetic flux $\Phi_m = \iint_{\text{area enclosed by a path } \Gamma} \vec{B} \cdot d\vec{a}$ to show that $\vec{A} = \frac{1}{2} \mu_0 n r I \vec{e}_\varphi$ inside the solenoid and $\vec{A} = \frac{1}{2} \mu_0 n \frac{b^2}{r} I \vec{e}_\varphi$ outside the solenoid. Here, $\vec{A}$ is the vector potential, b is the radius of ultrathin solenoid and r is the distance from the long central axis. Considering the solution for the vector field outside the solenoid: Is there a magnetic field outside the solenoid?

Exercise 3.

(Ring falling towards a wire / Category I)

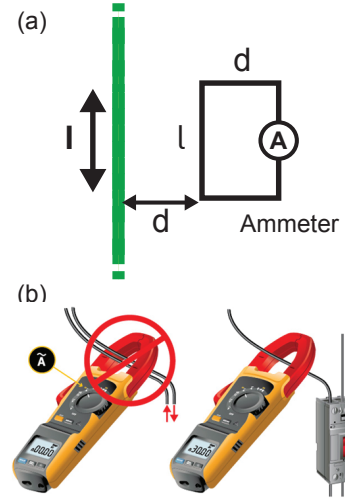
Consider a conducting ring falling towards a conducting wire carrying a current i , see figure. According to the induction law, a current i_{ind} will be induced in the ring itself. In what direction?



Exercise 4.**(Current clamp / Category I)**

Imagine a straight electric power cable (green line in sketch (a)) which is plugged into the electric grid ($I = I_0 \sin \omega t$ with $\omega = 2\pi f$). The frequency f in the electric grid amounts to 50 Hz. You want to find out the current flowing in the cable without removing the isolation. One possibility is to put a wire loop next to the cable and detect the induced current.

- Calculate the electromotive force emf in a rectangular wire loop with length $l = \frac{1}{\ln(2)}$ m and width d , which is placed next to the cable at distance d as indicated in sketch (a).
- The wire has a resistance of $R = 1 \Omega$. In the wire loop, a peak current of $I_{peak} = 1$ mA is measured with an ammeter. Calculate the current flowing inside the power cable.
- The same working principle is also used in commercial devices. Sketch (b) shows an excerpt from a manual of a current clamp. Can you explain the instructions? Suppose the device works exactly as in the previous question, and that whatever inside the clamp plays the role of the green wire.
- Does a current clamp based on induction also work for a DC current?

**Exercise 5.****(Vector potential and magnetic field for a circular current loop/Advanced Category III)**

Consider the problem of a circular loop of radius R , lying in the x, y -plane, centered at the origin, and carrying a current I , as shown in the figure.

- Calculate the vector potential $\vec{A}(r, \theta, \phi)$ for $r \gg R$ at point P in the x, z -plane ($\phi = 0$) as sketched. What is the direction of $\vec{A}$ following your calculation. Can you explain the orientation with symmetry arguments.
Hint: $(r^2 + R^2 - 2rR \sin \theta \cos \phi')^{-1/2} \cong \frac{1}{\sqrt{r^2 + R^2}} \left(1 + \frac{rR}{r^2 + R^2} \sin \theta \cos \phi' \right)$
- How does $\vec{A}$ vary as a function of ϕ at fixed r and θ ? For this consider the direction of $\vec{A}$ when P is either in the x, z -plane ($\phi = 0$) as sketched or assumed to be in the y, z -plane ($\phi = \pi/2$). Does this variation as a function of ϕ agree with the symmetry arguments?
- Show that the result of part (a) can be expressed as $\vec{A}(r, \theta, \phi) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$ where, $\vec{m} = I\vec{a}$ is the *magnetic dipole moment* of the current loop and the vector $\vec{a}$ is the directed area of the region surrounded by the loop.
- Calculate the magnetic field $\vec{B}(r, \theta, \phi)$ for $r \gg R$ from $\vec{A}(r, \theta, \phi)$.

