Exercise sheet 4: Capacitors, Capacitance

2/10/2024

We indicate the challenges of the problems by categories II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.

(Gauss's Law with a Cube) (Category I)

A point charge with charge Q is located at the corner of a cube (Fig. 1). Find the flux through the shaded area (Fig. 1) in terms of Q/ϵ_0.

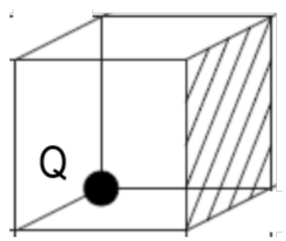

Figure 1: The point-like charge Q is exactly at the corner of a cube.

Exercise 2.

(Gauss's Law with Spheres and Shells) (Category I)

Consider the following four configurations (Fig. 2): (i) a point charge (ii) a sphere of radius r_0 with constant charge density ρ (iii) a sphere of radius r_0 with inhomogeneous charge density $\rho = \rho(r)$ and (iv) a thin spherical shell of radius r_0 with constant charge density ρ. For each case, a total charge Q is considered. Find the electric field at point r_1 in terms of Q and r_1 for each configuration. Assume $r_1 > r_0$.

Exercise 3.

(Application of the Laplace equation/Category II)

We consider two concentric spherical shells made of a conductor with negligible thickness. The inner shell has a radius R_1, the outer one a radius R_2. The inner shell is at potential $\phi_1 > 0$ while the outside one is connected to the ground with a conductor wire, i.e., $\phi_2 = 0$. The potential at infinity is chosen to be zero. To calculate the electric field in such a scenario, one uses the so-called Laplace equation $\vec{\nabla}^2\phi = 0$. This equation is consistent with the Coulomb law and needs to be integrated by considering the boundary conditions.

a) Make a drawing of the problem. Integrate the Laplace's equation between the two spherical shells to find the potential $\phi(\vec{r})$. Draw this potential.

b) Find the electric field from $\phi(\vec{r})$ and draw appropriate vectors.

c) Find the total charge in each sphere.

d) The capacitance is defined as $C = \frac{Q}{\phi_1 - \phi_2}$. Calculate C.

Hint: Look into Mathematical Tool Box 1, look at the functional form of $\nabla^2 f$ and integrate over the relevant coordinate.

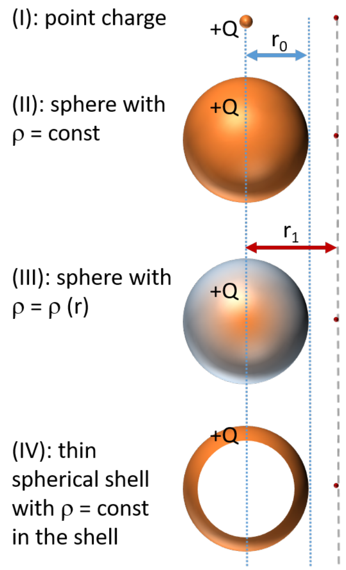

Figure 2: Four configurations, each carrying a total charge Q.

Exercise 4.
(Energy density and capacitance of cylindrical coordinate/Category II; 20 mins are expected for the solution after training for the written exam)
Consider two infinitely long concentric conductor cylinders with radius a and b. They form a capacitor: The charge on the inner cylinder is $+Q$. By choosing the adequate coordinate system:

a) Find the electric field $\vec{E}$ between the two conductors as a function of the relevant charge density σ.

b) Calculate the potential difference between the two cylinders.

c) Calculate the total energy per length, U/L, using $\frac{U}{L} = \frac{1}{L}\int \frac{\varepsilon_0|\vec{E}|^2}{2} dV$.

d) What is the capacitance per length, C/L?

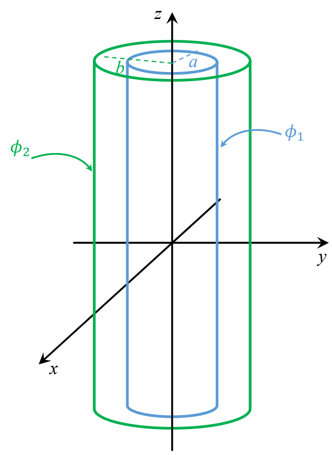

Figure 3: Sketch of the cylindrical equipotential surfaces.