Exercise sheet 3: Fields and Potentials, Gauss's Law

25/09/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

For exercise 1 it is instructive to first watch the video (link) and slides containing the solution for an electric field of a charged half-sphere. There we outline how to first analyze symmetries in a 3D scenario and second perform an integration considering ring-like elements.

Exercise 1.
(Potential and Electric Field of a Charged Circle) (Category I)

a) Find the expression for the potential ϕ at a height z over the center of a circle consisting of a uniformly charged line as shown in Fig. 1. The general formula for the potential is $\phi(\vec{r}) = \int \frac{\rho(\vec{r'})dV}{4\pi\epsilon_0|\vec{r}-\vec{r'}|}$. Hint: Consider that the line is a 1D charge distribution.

b) Find the electric field at the same point.

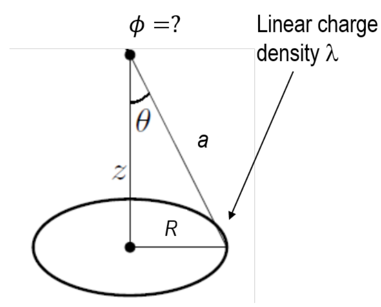

Figure 1: Only the line contains the charges. The black points indicate the center of the loop (bottom) and the position (top) at which the potential ϕ needs to be evaluated.

Exercise 2.
(Charge Density from Electric field) (Category I)

We consider an electric field given by $\vec{E}(\vec{r}) = kr^3\hat{r}$ in spherical coordinates. k is a positive constant and $\hat{r}$ is the unit vector.

a) Which units does the constant k have?

b) Find the expression for the corresponding charge density ρ. How does it depend on the radial distance r?

c) Sketch $\rho(r)$ as a function of radial distance along a line passing through the origin and in a plane intersecting the origin.

d) How large is ρ at $r = 1$ cm if the electric field amounts to $E = 5$ kV/m at the same position?

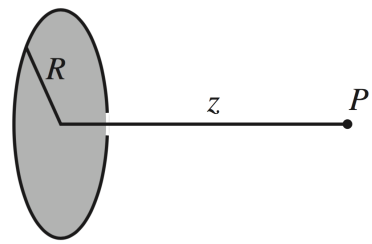

Figure 2: Sketch of a charged disk.

Exercise 3.
(Charged Disk) (Category II)
We consider a disk of radius R, carrying a total charge Q, uniformly spread over the disk (Fig. 2).

a) Calculate the electric field at a distance z along the disk's axis.

b) By considering the $R \to \infty$ limit, find the electric field generated by a charged infinite plane. Hint: For a charged infinite plane, the surface charge density σ can be assumed finite.

c) When R is finite, discuss the $z \gg R$ and $z \ll R$ limits. Hint: consider the electric field in the given limiting cases up to leading order.

Exercise 4.
(Non-uniformly charged sphere/Category II)

An insulating solid sphere with radius R is unevenly charged. The positive charge density is described by $\rho = Kr$ where K is a positive constant (with units of $\mathrm{C/m^4}$) and r is the distance (in units of m) from center of the sphere.

a) Assume the sphere to be surrounded by vacuum (Fig. 3). Find the equation describing the magnitude of the electric field E at a distance r from the center of the sphere in terms of constant K and radius R. Consider both cases of $r < R$ and $r > R$. (Hint: the spherical symmetry allows for Gauss's law with appropriately chosen Gaussian surfaces.)

b) Sketch the result $E(r)$ as a function of r from $r = 0$ to $r > R$.

c) Determine the equation for the electric potential function $\phi(r)$ as a function of r in terms of constant K and radius R. Sketch the result and provide the solutions for $\phi(r)$ at $r = 0$ and $r = R$.

d) A point-like negative test charge q with mass m is positioned at a distance $r = 4R$ and first held at rest. Then it is released. Find the equation for the velocity v at $r = 2R$ in terms of K, R, q, and m.

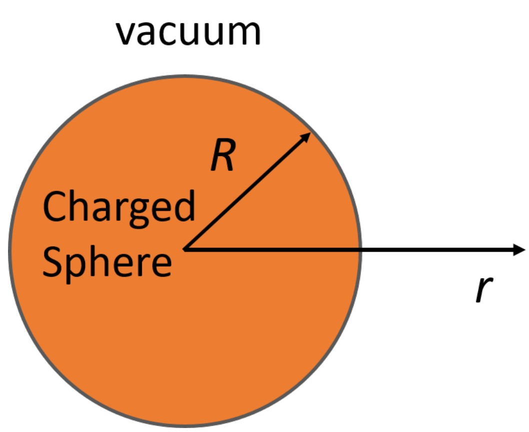

Figure 3: Sketch of the charged sphere with vacuum surrounding.