# Exercise sheet 14: Reflection, refraction, interaction of EM waves with matter, Poynting vector, Gauss Law, Magnetic fields, Ampère's Law

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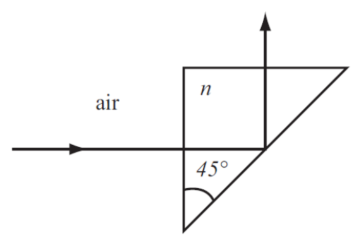
18/12/2024

We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

**Exercise 1.**
**(EM wave in an absorbing material (good conductor)/Category I)**

Consider an electromagnetic wave (EM) $\vec{E}(\vec{x},t) = E_0\hat{y} \cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ propagating along $\vec{x}$-direction in a *lossy* material. We assume metallic aluminium (Al) exhibiting a complex refractive index $\tilde{n} = n + i\kappa$ with $\kappa > 0$. In a lossy material $\vec{k} = \frac{\omega\tilde{n}}{c}\hat{k}$ with $\hat{k}$ being the unit vector in propagation direction. We consider light that in vacuum has a wavelength $\lambda$ of 546 nm. For this one finds $n = 0.82$ and $\kappa = 5.99$ in Al. Show that the light wave exhibits a decaying amplitude when entering Al under normal incidence and quantify the so-called decay length after which the intensity diminishes to $1/e$. Compare this value with the wavelength.

**Exercise 2.**
**(Refraction and (total internal) reflection/Category II**
**(After training for solution: 10 min))**




In the figure we sketch a situation where no light beam leaves a prism at the right edge (= total internal reflection at the second surface that the light hits). The refractive index $n$ is such that the angle of the refracted beam at the right surface is just 90°. This is the definition of the critical angle $\theta_c$ for total internal reflection.

a) How large is $n$ of the prism assuming that there is air outside the prism with a refractive index equal to 1?

b) What kind of material would exhibit such a value $n$?

c) Where does the incident light beam go when the index $n$ is (i) doubled, and (ii) halved? Calculate the refraction angle when refracted light is expected.

**The following four problems review topics of the previous weeks/chapters in a way which is consistent with a written exam problem**

**Exercise 3.**
**(Poynting vector concept applied to current in a perfect coaxial cable/Category II (After training for solution: 20 min))**

A coaxial cable consists of two concentric long hollow cylinders of zero resistance (perfect conductors); the inner has a radius $a$, the outer has radius $b$, and the length of both is $l$, with $l \gg b$, as shown in Fig. 1. The cable transmits power from a battery to a load via a DC current $I$. The battery provides an electromotive force $\varepsilon$ between the two conductors at one end of the cable, and the load is a resistance $R$ connected between the two perfect conductors at the other end of the cable.

a) We consider that the battery charges the inner conductor to a charge -$Q$ and the outer conductor to a charge $+Q$ due to the potential difference applied to them. At the same time a current $I$ flows down the inner conductor and back up the outer one as power is dissipated in $R$. Find the direction and magnitude of the electric field $\vec{E}$ everywhere.

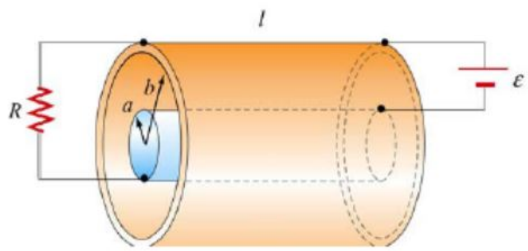



Figure 1: A coaxial cable consists of two concentric long hollow cylinders of zero resistance.

b) Find the direction and magnitude of the magnetic field $\vec{B}$ everywhere.

c) Apply the concept of the Poynting vector $\vec{S}$ inside the coaxial cable and calculate $\vec{S}$.

d) By integrating $\vec{S}$ over an appropriate surface, find the power that flows into the coaxial cable.

e) How does your result in (d) compare to the power dissipated in the resistor?

**Exercise 4.**
**(Loop on string/Category II (taken from an exam) time: 25 min )**
Consider a current-carrying wire with a constant current $I_1$ that is infinitely long. The current is along $y$-direction. A square loop formed by a massless rigid conductor is positioned symmetrically above the wire as sketched. The loop is parallel to the $x, y$-plane. The loop carries a constant current $I_2$ and resides on a rigid string that is at height $z = h$ and parallel with the wire. The loop can slide along and rotate around the nonconducting string without friction. The directions of currents flowing in the closed conductor loop and in the wire are indicated by arrows.

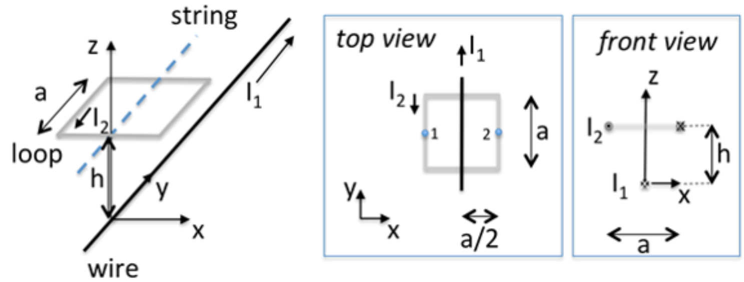



Figure 2: Rectangular loop made of a perfect conductor (grey) positioned on a string (broken line). Separated by a height h a current-carrying wire (black line) is present. The wire is straight and infinitely long. In the center a top view is shown indicating the two positions for analyzing forces and torques. Geometrical parameters and current-flow directions are defined in the central sketch and on the right image which shows a front view.

a) Calculate forces and torques acting at positions 1 and 2 due to current $I_1$ (see figure) as a function of separation $h$ and side length $a$.

b) What is the total force on the string?

c) Does the loop rotate? If yes, what is the sense of rotation?

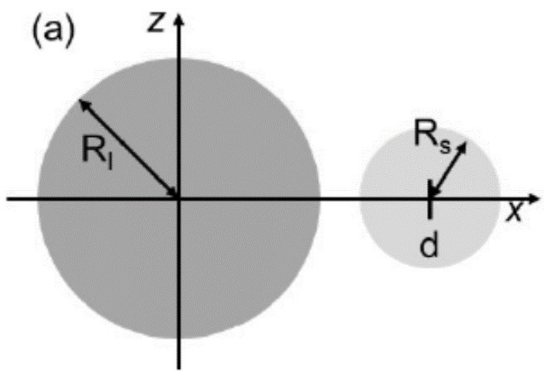



Figure 3: A large solid sphere positioned at the origin of the $x, y, z$ coordinate system, and a small solid sphere located at $x = d$. We show the side view onto the central cross-sectional plane of the charged solid spheres (see text).

**Exercise 5.**

**(Charged Spheres) (Category II time needed after training: 20 min; this is taken from an exam)**

Positive electrical charges are uniformly distributed in the volume of a large three-dimensional sphere of radius $R_\mathrm{l}$. The value of the uniform charge density is $\rho_\mathrm{C}$. Assume that the large sphere of radius $R_\mathrm{l}$ is positioned next to a small sphere [Fig. 3 (a)]. The central coordinate of the former sphere is at the origin of the coordinate system. The central coordinate of the small sphere is on the $x$-axis with $x = d$. The small sphere of radius $R_\mathrm{s}$ is uniformly charged with negative charges of a density $\rho_\mathrm{s} = -\rho_\mathrm{C}$. The radii of the two spheres are such that they do not touch. Derive the formula for the electrical field vector $\vec{E}$ on the $x$-axis *inside* the small sphere depending on $\rho_\mathrm{C}, R_\mathrm{l}, x, \text{and}\, d$.

**Exercise 6.**
**(Optical Fiber) (Category II time needed after training: 15 min; this is taken from an exam)**

We consider a cylindrical optical fiber (Fig. 4) consisting of lossless transparent materials A in the core and B in the shell (cladding). The fiber resides in vacuum. The shell of material B surrounds the core. Light of a specific wavelength enters the fiber as sketched in the figure. The materials have different indices of refraction $n_{\mathrm{A}} = 1.480$ and $n_{\mathrm{B}} = 1.440$. All light rays shown are in the same plane which is the mirror plane of the fiber through the central axis.

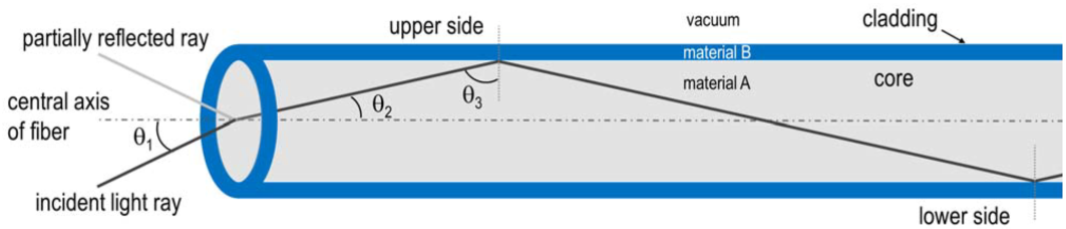



Figure 4: Sketch of the fiber.

a) What is the critical angle $\theta_3$ for the total internal reflection at the interface between materials A and B, i.e., core and cladding?

b) For what range of angle $\theta_1$ is light totally internally reflected at the core-cladding interface?

c) If light is totally internally reflected at the upper core-cladding interface of the fiber, will it be totally internally reflected at the lower core-cladding interface (assuming the relevant interfaces to be parallel)?