Exercise sheet 13: Propagating and standing waves, Poynting vector, superposition

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We indicate the challenges of the problems by categories I ("warming-up"), II ("exam-level"), III ("advanced"). For your orientation: problems attributed to category II have been or could have been considered for an exam (assuming a specific duration for finding the solution; see comments in the solutions). The exact problem setting cannot be repeated in an exam however.

Exercise 1.

(Radio station/Category II (After training for solution: 25 min))

A radio station (rs) is allowed to broadcast at a maximum average power of 25 kW radially. If an electric field amplitude of 0.020 V/m is considered to be acceptable for receiving the radio transmission with a relevant signal strength, estimate how many kilometers away you might be able to hear this station in your radio. Assume a point-like source which emits a spherical wave. Integrate $\vec{S}$ over an appropriately chosen surface.

Exercise 2.

(Poynting vector in a capacitor/Category II (After training for solution: 20 min))

a) Show that the so-called Poynting vector $\vec{S} = \frac{1}{\mu_0}(\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t))$, which describes the energy flux density, points radially inwards toward the center of a circular parallel–plate capacitor when it is being charged [this means: $E = E(t)$].

b) Integrate $\vec{S}$ over the cylindrical boundary of the capacitor gap to obtain the energy flux and show that the rate at which energy enters the capacitor via the Poynting vector $\vec{S}$ is equal to the rate at which electrostatic energy U is being stored in the electric field of the capacitor. Ignore fringing fields of $\vec{E}$ to show $-\oiint \vec{S} \cdot d\vec{a} = \frac{dU}{dt}$.

Exercise 3.

(Energy flow for a standing wave/Category II (After training for solution: 35 min))

Consider the standing electromagnetic wave from the lecture given by $\vec{E} = 2\hat{z}E_0[\sin(ky)\cos(\omega t)]$ and $\vec{B} = -2\hat{x}\frac{E_0}{c}[\cos(ky)\sin(\omega t)]$.

a) Calculate the time-dependent energy densities $u_E(y,t)$ and $u_B(y,t)$. Draw plots of the densities at ωt values of 0, $\pi/4$, $\pi/2$, and $3\pi/4$.

b) Calculate the y component of the time-dependent Poynting vector, $S_y(y,t)$, and plot its value at different times corresponding to ωt equal to 0, $\pi/4$, $\pi/2$ and $3\pi/4$. Are these plots consistent with how the energy densities vary as a function of time t?

c) How large is the time-averaged Poynting vector?

Exercise 4.

(Confined waves/Category II (After training for solution: 15 min))

Consider a string with linear mass density σ, length L and tension T. We neglect gravitational force on the string. The string is along the $\hat{x}$ direction, starting at $x = 0$, ending at $x = L$. We consider small deformations $\xi(x,t)$ along the string. The ends of the string are rigidly fixed to an extremely heavy wall. Consider the wave equation obeyed by $\xi(x,t)$ as discussed in the lecture. The general solution to this equation is the superposition of left- and right-propagating waves. We assume that all of those waves have the same amplitude ξ_0 and phase velocity v.

a) Explain why the boundary conditions for $\xi(x,t)$ impose that the wavelength of left- and right-propagating waves must be of the form $\lambda_i = \frac{2L}{i}$, $i = 1, 2, ...$, in order to lead to constructive interference. What are the corresponding wavevectors k_i and frequencies ω_i? The numbers i count the so-called harmonics.

b) Show that the solution $\xi_i(x,t) = 2\xi_0 \sin(k_i x)\cos(\omega_i t)$ fulfills the wave equation and boundary conditions.

c) Consider a guitar with a steel string fixed rigidly between two points. When excited, it has a number of frequencies ω_i (harmonics) at which it vibrates. Assume that the tension put on the guitar string produces a phase velocity of 470 m/s. The length of the string is 66.5 cm. What are the frequencies $\nu_i = \omega_i/(2\pi)$ of the first ($i = 1$) and second ($i = 2$) harmonic, i.e. the lowest and the second lowest frequencies? Can their sound be heard?